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On Muckenhoupt–Wheeden Conjecture ☆

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Abstract

Let M denote the dyadic Maximal Function. We show that there is a weight w, and Haar multiplier T for which the following weak-type inequality fails

$$\sup_{t>0} t w\big(\big\{x \in \mathbb{R}\colon \big|Tf(x)\big| > t\big\}\big) \leqslant C \int_{\mathbb{R}} |f| Mw(x)\, dx.$$

(With T replaced by M, this is a well-known fact.) This shows that a dyadic version of the so-called Muckenhoupt–Wheeden Conjecture is false. This accomplished by using current techniques in weighted inequalities to show that a particular L^2 consequence of the inequality above does not hold.

Keywords: Singular integrals; Weights; Calderón–Zygmund operators

1. Introduction

The starting point of this work goes back to 1971 [7], when C. Fefferman and E. Stein proved that if w is a weight, namely a non-negative locally integrable function, and M denotes the Hardy–Littlewood maximal operator then

$$\sup_{t>0} t w\big(\big\{x \in \mathbb{R}^n\colon\ Mf(x) > t\big\}\big) \leqslant c \int_{\mathbb{R}^n} |f| Mw(x).$$

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A very natural question was then raised by B. Muckenhoupt and R. Wheeden (see [13]): could we replace the Hardy–Littlewood maximal operator M by a Calderón–Zygmund operator? Their conjecture, known as the Muckenhoupt–Wheeden Conjecture, is stated below.

Conjecture 1.1 *(Muckenhoupt–Wheeden). Let w be a weight and M be the Hardy–Littlewood maximal operator. Let T be a Calderón–Zygmund operator with $\|T\|_{CZO} \leqslant 1$. Then*

$$\sup_{t>0} t w\big(\{x \in \mathbb{R}: \ |Tf(x)| > t\}\big) \leqslant C \int_{\mathbb{R}} |f| Mw(x)\, dx. \tag{1.2}$$

The exact definition of Calderón–Zygmund operator need not concern us here, though it certainly includes the non-positive Hilbert transform (see Chapter VII of [18] for precise definitions). The hope was that the conjecture identified a somewhat robust principle. We herein disprove the *dyadic version* of this conjecture. So, M is replaced by the (smaller) dyadic maximal function, and T will be a Haar multiplier, which are the simplest possible dyadic Calderón–Zygmund operators.

Endpoint estimates are known to be the most delicate ones, and very frequently they are also the most powerful. That is the case of the Muckenhoupt–Wheeden Conjecture. For instance, an extrapolation result due to D. Cruz-Uribe and C. Pérez [5] shows this: If w is a weight and (1.2) holds with T a sublinear operator then

$$\int_{\mathbb{R}} \big|T(f)\big|^p w(x)\, dx \leqslant \int_{\mathbb{R}} |f|^p \left(\frac{Mw}{w}\right)^p w(x)\, dx. \tag{1.3}$$

The dyadic version of this result is also true (see [5], Remark 1.5).

With a few partial results that we shall discuss later in the introduction, the Muckenhoupt–Wheeden Conjecture has been open up to today's date. In this paper, we answer the dyadic version of (1.2) in the negative by disproving (1.3). We are ready to state our main theorem.

Theorem 1.4. *There exist a weight w and a Haar multiplier T for which T is unbounded as map from $L^2((\frac{Mw}{w})^2 w)$ to $L^2(w)$.*

As a corollary we solve a long-standing conjecture.

Corollary 1.5. *The Muckenhoupt–Wheeden Conjecture in its dyadic version is false.*

For the proof we construct a measure w and a Haar multiplier T that avoids all cancellations. The tool behind this construction is a corona decomposition, which has proven to be very useful in finding sharp estimates when the weight is in the A_p class, see [12,10,9,11].

There has been evidence for both positive and negative answers to the conjecture. S. Chanillo and R. Wheeden [3] showed that a square function satisfied the Muckenhoupt–Wheeden Conjecture. We also mention the work of Buckley [1], who in dimension n proved that (1.2) holds for weights $w_\delta(x) = |x|^{-n(1-\delta)}$ for $0 < \delta < 1$.