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Classification of the centers and their isochronicity for a class of polynomial differential systems of arbitrary degree

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Abstract

In this paper we classify the centers localized at the origin of coordinates, and their isochronicity for the polynomial differential systems in $\mathbb{R}^2$ of degree d that in complex notation $z = x + iy$ can be written as

$$\dot{z} = (\lambda + i)z + Az^{(d-n+1)/2}\bar{z}^{(d+n-1)/2} + Bz^{(d+n+1)/2}\bar{z}^{(d-n-1)/2} + Cz^{(d+1)/2}\bar{z}^{(d-1)/2} + Dz^{(d-(2+j)n+1)/2}\bar{z}^{(d+(2+j)n-1)/2},$$

where j is either 0 or 1. If $j = 0$ then $d \geqslant 5$ is an odd integer and n is an even integer satisfying $2 \leqslant n \leqslant (d+1)/2$. If $j = 1$ then $d \geqslant 3$ is an integer and n is an integer with converse parity with d and satisfying $0 < n \leqslant [(d+1)/3]$ where $[\cdot]$ denotes the integer part function. Furthermore $\lambda \in \mathbb{R}$ and $A, B, C, D \in \mathbb{C}$. Note that if $d = 3$ and $j = 0$, we are obtaining the generalization of the polynomial differential systems with cubic homogeneous nonlinearities studied in K.E. Malkin (1964) [17], N.I. Vulpe and K.S. Sibirskii (1988) [25], J. Llibre and C. Valls (2009) [15], and if $d = 2$, $j = 1$ and $C = 0$, we are also obtaining as a particular case the quadratic polynomial differential systems studied in N.N. Bautin (1952) [2], H. Zoladek (1994) [26]. So the class of polynomial differential systems here studied is very general having arbitrary degree and containing the two more relevant subclasses in the history of the center problem for polynomial differential equations.

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1. Introduction and statement of the main results

One of the main problems in the qualitative theory of real planar polynomial differential systems is the center–focus problem; i.e. to distinguish when a singular point is either a focus or a center. The notion of *center* goes back to Poincaré in [20]. He defined it for a vector field on the real plane; i.e. a singular point surrounded by a neighborhood filled with closed orbits with the unique exception of the singular point.

The classification of the centers of the polynomial differential systems started with the quadratic ones with the works of Dulac [7], Kapteyn [11,12], Bautin [2] and others. Schlomiuk, Guckenheimer and Rand in [24] described a brief history of the problem of the center in general, and it includes a list of 30 papers covering the topic and the history of the center for the quadratic case (see pages 3, 4 and 13). Here we are mainly interested in finding new families of centers of polynomial differential systems of arbitrary degree and in studying their cyclicity and isochronicity. There are other interesting problems related with the centers that in this paper we do not consider as for instance, their phase portraits in the Poincaré disc, or the kind of first integrals that the centers can have, or the bifurcation diagram of the different phase portraits of centers in the parameter space, etc. In the case of quadratic centers these last problems were studied by several authors, see for instance Schlomiuk [22,23] and the references therein.

In this paper we consider the polynomial differential systems in the real (x, y)-plane which has a singular point at the origin with eigenvalues $\lambda \pm i$ and that can be written as

$$\begin{aligned}\dot{z} &= (\lambda + i)z + Az^{(d-n+1)/2}\bar{z}^{(d+n-1)/2} + Bz^{(d+n+1)/2}\bar{z}^{(d-n-1)/2} \\ &\quad + Cz^{(d+1)/2}\bar{z}^{(d-1)/2} + Dz^{(d-(2+j)n+1)/2}\bar{z}^{(d+(2+j)n-1)/2},\end{aligned} \tag{1}$$

where j is either 0 or 1. If $j = 0$ then $d \geqslant 5$ is an odd integer and $n \geqslant 2$ is an even integer satisfying $n \leqslant (d+1)/2$. If $j = 1$ then $d \geqslant 3$ is an integer and $n > 0$ is an integer with converse parity with d and satisfying $n \leqslant [(d+1)/3]$. Furthermore $\lambda \in \mathbb{R}$, and $A, B, C, D \in \mathbb{C}$. When $j = 0$ we are considering the polynomial differential systems

$$\begin{aligned}\dot{z} &= (\lambda + i)z + Az^{(d-n+1)/2}\bar{z}^{(d+n-1)/2} + Bz^{(d+n+1)/2}\bar{z}^{(d-n-1)/2} \\ &\quad + Cz^{(d+1)/2}\bar{z}^{(d-1)/2} + Dz^{(d-2n+1)/2}\bar{z}^{(d+2n-1)/2},\end{aligned}$$

and when $j = 1$ we are considering the polynomial differential systems

$$\begin{aligned}\dot{z} &= (\lambda + i)z + Az^{(d-n+1)/2}\bar{z}^{(d+n-1)/2} + Bz^{(d+n+1)/2}\bar{z}^{(d-n-1)/2} \\ &\quad + Cz^{(d+1)/2}\bar{z}^{(d-1)/2} + Dz^{(d-3n+1)/2}\bar{z}^{(d+3n-1)/2}.\end{aligned}$$

The vector field associated to system (1) is formed by the linear part $(\lambda + i)z$ and by a homogeneous polynomial of degree d formed by four monomials in complex notations. Since the