

# Pointwise symmetrization inequalities for Sobolev functions and applications <sup>☆</sup>

Joaquim Martín <sup>a,1</sup>, Mario Milman <sup>b,\*</sup>

<sup>a</sup> *Department of Mathematics, Universitat Autònoma de Barcelona, Spain*

<sup>b</sup> *Department of Mathematics, Florida Atlantic University, United States*

Received 12 August 2009; accepted 23 February 2010

Available online 30 March 2010

Communicated by Luis Caffarelli

---

## Abstract

We develop a technique to obtain new symmetrization inequalities that provide a unified framework to study Sobolev inequalities, concentration inequalities and sharp integrability of solutions of elliptic equations.

© 2010 Elsevier Inc. All rights reserved.

**Keywords:** Logarithmic Sobolev inequalities; Poincaré; Symmetrization; Isoperimetric inequalities; Concentration

---

## Contents

|   |     |
|---|-----|
| 1. Introduction . . . . .                                     | 122 |
| 2. Background . . . . .                                       | 129 |
| 2.1. Rearrangement invariant spaces . . . . .                 | 131 |
| 3. Symmetrization using truncation and isoperimetry . . . . . | 135 |
| 4. Pólya–Szegő . . . . .                                      | 144 |
| 4.1. Model Case 1: log concave measures . . . . .             | 144 |
| 4.2. Model Case 2: the $n$ -sphere . . . . .                  | 147 |

---

<sup>☆</sup> This paper is in final form and no version of it will be submitted for publication elsewhere.

<sup>\*</sup> Corresponding author.

*E-mail addresses:* [jmartin@mat.uab.cat](mailto:jmartin@mat.uab.cat) (J. Martín), [extrapol@bellsouth.net](mailto:extrapol@bellsouth.net) (M. Milman).

*URL:* <http://www.math.fau.edu/milman> (M. Milman).

<sup>1</sup> Supported in part by Grants MTM2007-60500, MTM2008-05561-C02-02.

|         |  |     |
|---------|--|-----|
| 4.3.    | Model Case 3: Model Riemannian manifolds . . . . .                 | 149 |
| 5.      | Poincaré inequalities . . . . .                                    | 150 |
| 5.1.    | Poincaré inequalities for the model cases . . . . .                | 155 |
| 6.      | Poincaré inequalities and Cheeger's inequality . . . . .           | 159 |
| 6.1.    | Poincaré inequalities and Hardy operators . . . . .                | 159 |
| 6.2.    | Isoperimetric Hardy type . . . . .                                 | 160 |
| 7.      | Transference principle . . . . .                                   | 166 |
| 7.1.    | Gaussian isoperimetric type and a question of Triebel . . . . .    | 169 |
| 8.      | Estimating isoperimetric profiles via semigroups . . . . .         | 172 |
| 9.      | Higher order Sobolev inequalities . . . . .                        | 175 |
| 10.     | Integrability of solutions of elliptic equations . . . . .         | 180 |
| 10.1.   | Sharpness of the results . . . . .                                 | 189 |
| 10.1.1. | Between exponential and Gaussian measure . . . . .                 | 190 |
| 11.     | Connection with some capacity inequalities due to Maz'ya . . . . . | 193 |
|         | Acknowledgments . . . . .  | 194 |
|         | Appendix A. A few (and only a few) bibliographical notes . . . . . | 195 |
|         | References . . . . .   | 196 |

---

## 1. Introduction

Symmetrization is a very useful classical tool in PDE's and the theory of Sobolev spaces. The standard symmetrization inequalities, like many other inequalities in the theory of Sobolev spaces, are often formulated as norm inequalities. One drawback is that these inequalities need to be (re)proven separately for different classes of spaces (e.g.  $L^p$ , Lorentz, Orlicz, Lorentz–Karamata, etc.). For this purpose interpolation can be a useful tool, but one may lose information in the extreme cases. Moreover, the end point Sobolev embeddings usually require a different type of spaces (often called “extrapolation spaces”). Thus, for example, the optimal embeddings of  $L^p$  based Sobolev spaces on  $n$ -dimensional Euclidean space are the Lorentz  $L(p^*, p)$  spaces, where  $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ ,  $1 \leq p < n$ , but for the limiting case  $p = n$  it is necessary to replace the Lorentz norms by suitable variants in order to accommodate exponential integrability. One way to deal with this problem is to use pointwise rearrangement inequalities; among the many contributions in this direction here we only mention just a few [56,118,117,70,9,20,54,3,38,35,81,82,109,77,32], and refer the reader to the references therein. An added complication arises because different geometries produce different types of optimal spaces: a dramatic example is provided by Gaussian measure, where the optimal target spaces for the embeddings of  $L^p$  based Sobolev spaces are the  $L^p(\text{Log } L)^{p/2}$  spaces (cf. [58,53,1,18,19], and the references therein). Likewise, in the study of integrability of solutions of elliptic equations, the corresponding optimal results depend on the geometry. As a consequence, although many of the methods used in the treatment of the different cases are similar each case still requires a separate treatment.

In our recent work (cf. [90,86,87]) we have developed new symmetrization inequalities that address all these issues and can be applied to provide a unified treatment of sharp Sobolev–Poincaré inequalities, concentration inequalities and sharp integrability of solutions of elliptic equations. Our inequalities combine three basic features, each of which may have been considered before but, apparently, not all of them simultaneously; namely our inequalities are (i) pointwise rearrangement inequalities, (ii) incorporate in their formulation the isoperimetric profile and (iii) are formulated in terms of oscillations.

Download English Version:

<https://daneshyari.com/en/article/4666938>

Download Persian Version:

<https://daneshyari.com/article/4666938>

[Daneshyari.com](https://daneshyari.com)