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Pointwise symmetrization inequalities for Sobolev functions and applications ☆

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Abstract

We develop a technique to obtain new symmetrization inequalities that provide a unified framework to study Sobolev inequalities, concentration inequalities and sharp integrability of solutions of elliptic equations.

Keywords: Logarithmic Sobolev inequalities; Poincaré; Symmetrization; Isoperimetric inequalities; Concentration

Contents

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1. Introduction

Symmetrization is a very useful classical tool in PDE's and the theory of Sobolev spaces. The standard symmetrization inequalities, like many other inequalities in the theory of Sobolev spaces, are often formulated as norm inequalities. One drawback is that these inequalities need to be (re)proven separately for different classes of spaces (e.g. L^p, Lorentz, Orlicz, Lorentz–Karamata, etc.). For this purpose interpolation can be a useful tool, but one may lose information in the extreme cases. Moreover, the end point Sobolev embeddings usually require a different type of spaces (often called "extrapolation spaces"). Thus, for example, the optimal embeddings of L^p based Sobolev spaces on n-dimensional Euclidean space are the Lorentz $L(p^*, p)$ spaces, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, $1 \leqslant p < n$, but for the limiting case $p = n$ it is necessary to replace the Lorentz norms by suitable variants in order to accommodate exponential integrability. One way to deal with this problem is to use pointwise rearrangement inequalities; among the many contributions in this direction here we only mention just a few [56,118,117,70,9,20,54,3,38,35,81,82,109,77,32], and refer the reader to the references therein. An added complication arises because different geometries produce different types of optimal spaces: a dramatic example is provided by Gaussian measure, where the optimal target spaces for the embeddings of L^p based Sobolev spaces are the $L^p(\operatorname{Log} L)^{p/2}$ spaces (cf. [58,53,1,18,19], and the references therein). Likewise, in the study of integrability of solutions of elliptic equations, the corresponding optimal results depend on the geometry. As a consequence, although many of the methods used in the treatment of the different cases are similar each case still requires a separate treatment.

In our recent work (cf. [90,86,87]) we have developed new symmetrization inequalities that address all these issues and can be applied to provide a unified treatment of sharp Sobolev–Poincaré inequalities, concentration inequalities and sharp integrability of solutions of elliptic equations. Our inequalities combine three basic features, each of which may have been considered before but, apparently, not all of them simultaneously; namely our inequalities are (i) pointwise rearrangement inequalities, (ii) incorporate in their formulation the isoperimetric profile and (iii) are formulated in terms of oscillations.