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Inverse problems with partial data for a Dirac system: A Carleman estimate approach

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Abstract

We prove that the material parameters in a Dirac system with magnetic and electric potentials are uniquely determined by measurements made on a possibly small subset of the boundary. The proof is based on a combination of Carleman estimates for first and second order systems, and involves a reduction of the boundary measurements to the second order case. For this reduction a certain amount of decoupling is required. To effectively make use of the decoupling, the Carleman estimates are established for coefficients which may become singular in the asymptotic limit.

Keywords: Inverse problems; Dirac operator; Carleman estimates

1. Introduction

This article is concerned with the inverse problem of determining unknown coefficients in a Dirac system from measurements made on part of the boundary. A standard problem of this type is the inverse conductivity problem of Calderón [4], where the purpose is to determine the electrical conductivity of a body by making voltage to current measurements on the boundary.

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In mathematical terms, if γ is a smooth positive function in the closure of a bounded domain $\Omega \subseteq \mathbf{R}^n$, the boundary measurements are given by the Cauchy data set

$$C_\gamma = \big\{(u|_{\partial\Omega}, \gamma\partial_\nu u|_{\partial\Omega});\ \nabla\cdot(\gamma\nabla u) = 0 \text{ in } \Omega,\ u \in H^1(\Omega)\big\}.$$

Here $u|_{\partial\Omega}$ and $\gamma\partial_\nu u|_{\partial\Omega}$ are the voltage and current, respectively, on $\partial\Omega$, corresponding to a potential u satisfying the conductivity equation in Ω ($\partial_\nu u$ denotes the normal derivative). The inverse problem is to determine the conductivity γ from the knowledge of the Cauchy data set C_γ.

The inverse conductivity problem has been well studied, and major results include [1,21,28] which prove that C_γ determines γ in various settings. Less is known about the partial data problem, where one is given two sets $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$ and the boundary measurements are encoded by the set

$$C_\gamma^{\Gamma_1,\Gamma_2} = \big\{(u|_{\Gamma_1}, \gamma\partial_\nu u|_{\Gamma_2});\ \nabla\cdot(\gamma\nabla u) = 0 \text{ in } \Omega,\ u \in H^1(\Omega)\big\}.$$

There are two main approaches for proving that γ is determined by $C_\gamma^{\Gamma_1,\Gamma_2}$. The first approach, introduced in [3,16], uses Carleman estimates with boundary terms to control solutions on parts of the boundary. The result in [16] is valid in dimensions $n \geqslant 3$ and for small sets Γ_2 (the shape depending on the geometry of $\partial\Omega$), but assumes that Γ_1 has to be relatively large. The second approach [12] is based on reflection arguments and is valid when $n \geqslant 3$ and $\Gamma_1 = \Gamma_2$ and Γ_1 may be a small set, but it is limited to the case where $\partial\Omega \setminus \Gamma_1$ is part of a hyperplane or a sphere. Results similar to [12] but without the last restriction were recently proved for $n = 2$ in [11] and for the linearized problem in [8].

We are interested in inverse problems with partial data for elliptic linear systems. In the case of full data (that is, $\Gamma_1 = \Gamma_2 = \partial\Omega$), there is an extensive literature including uniqueness results for the Maxwell equations [25,26], the Dirac system [22,27], and the elasticity system [9,23,24]. However, it seems that partial data results for systems are more difficult to establish. The reflection approach is in principle more straightforward to extend to systems, and the recent work [5] gives a partial data result analogous to [12] for the Maxwell equations. As for the Carleman estimate approach, there is a fundamental problem since Carleman estimates for first order systems, such as the ones in [27], seem to have boundary terms which are not useful in partial data results.

In this paper, we prove a partial data result analogous to [16] for a Dirac system. To our knowledge this is the first such partial data result for a system. The proof is based on Carleman estimates, and it involves a reduction to boundary measurements for a second order equation. The corresponding boundary term is handled by a Carleman estimate for second order systems, designed to take into account the amount of decoupling present in the original equation. In the set where one cannot decouple, we need to use the first order structure as well. The Carleman estimates need to be valid for coefficients which may blow up in the asymptotic limit, in order to obtain sufficiently strong estimates for solutions on the boundary.

Let us now state the precise problem. We consider the free Dirac operator in $\mathbf{R}^3$, arising in quantum mechanics and given by the 4×4 matrix

$$P(D) = \begin{pmatrix} 0 & \sigma\cdot D \\ \sigma\cdot D & 0 \end{pmatrix}, \tag{1.1}$$