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A parabolic two-phase obstacle-like equation

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Abstract

For the parabolic obstacle-problem-like equation

$$\Delta u - \partial_t u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}},$$

where λ_+ and λ_- are positive Lipschitz functions, we prove in arbitrary finite dimension that the free boundary $\partial\{u > 0\} \cup \partial\{u < 0\}$ is in a neighborhood of each "branch point" the union of two Lipschitz graphs that are continuously differentiable with respect to the space variables. The result extends the elliptic paper [Henrik Shahgholian, Nina Uraltseva, Georg S. Weiss, The two-phase membrane problem—regularity in higher dimensions, Int. Math. Res. Not. (8) (2007)] to the parabolic case. There are substantial difficulties in the parabolic case due to the fact that the time derivative of the solution is in general not a continuous function. Our result is optimal in the sense that the graphs are in general not better than Lipschitz, as shown by a counter-example.

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1. Introduction

1.1. Background and main result

In this paper we study the regularity of the parabolic obstacle-problem-like equation

$$\Delta u - \partial_t u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \quad \text{in } (0,T) \times \Omega, \tag{1.1}$$

where $T < +\infty$, $\lambda_+ > 0$, $\lambda_- > 0$ are Lipschitz functions and $\Omega \subset \mathbf{R}^n$ is a given domain. The problem arises as limiting case in the model of temperature control through the interior described in [4, 2.3.2] as $h_1, h_2 \to 0$.

We are interested in the regularity of the free boundary $\partial\{u > 0\} \cup \partial\{u < 0\}$. As the one-phase case (i.e. the case of a non-negative or non-positive solution) is covered by classical results, and regularity of the set $\{u = 0\} \cap \{\nabla u \neq 0\}$ can be obtained via the implicit function theorem (see Section 7 for higher regularity), the research focuses on the study of $\partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$.

In the stationary case—the two-phase membrane problem—the authors proved ([12] and [11]) that the free boundary $\partial\{u > 0\} \cup \partial\{u < 0\}$ is in a neighborhood of each branch point, i.e. a point in the set $\Omega \cap \partial\{u > 0\} \cap \partial\{u < 0\} \cap \{\nabla u = 0\}$, the union of (at most) two C^1-graphs. Note that the definition of "branch point" does not necessarily imply a bifurcation as that in Fig. 1. We formulate the main result in this paper.