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Relative K-stability and modified K-energy on toric manifolds

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Abstract

In this paper, we give a sufficient condition for both the relative K-stability and the properness of modified K-energy associated to Calabi's extremal metrics on toric manifolds. In addition, several examples of toric manifolds which satisfy the sufficient condition are presented.

MSC: primary 53C25; secondary 32J15, 53C55, 58E11

Keywords: Toric manifolds; Relative K-stability; Modified K-energy

0. Introduction

The relationship between various geometric stabilities and the existence of Kähler–Einstein metrics, more general, of Calabi's extremal metrics has been recently studied extensively [21, 6,7,13,14], etc. The goal is to establish necessary and sufficient condition for the existence of extremal metrics in the sense of Geometric Invariant Theory [26,21]. More recently, S.K. Donaldson studied a relationship between the K-stability of Tian–Donaldson and Mabuchi's K-energy on polarized toric manifolds, and in particular he proved that the K-energy is bounded from below under the assumption of K-stability in the case of toric surfaces [7]. In this paper, we want to generalize and improve some techniques in [7] to give a sufficient condition for both the K-stability

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and the properness of K-energy in the sense of polytopes associated to toric manifolds. In fact, we can discuss the relative K-stability (and the modified K-energy) associated to an extremal holomorphic vector field on M. This relative K-stability based on a modified Futaki invariant is a generalization of the K-stability and removes the obstruction arising from the non-vanishing of Futaki invariant [20].

An n-dimensional polarized toric manifold M corresponds to an integral polytope P in $\mathbb{R}^n$ which is described by an intersection of some half-spaces,

$$\langle l_i, x\rangle < \lambda_i, \quad i = 1, \ldots, d. \tag{0.1}$$

Here l_i are d vectors in $\mathbb{R}^n$ with all integral components, which satisfy the Delzant condition [1]. Without loss of generality, we may assume that the original point 0 lies in P, so all $\lambda_i > 0$.

Let X be an extremal holomorphic vector field associated to the polarized Kähler class on M[2] and $\|\theta_X\|_+$ be an invariant of potential θ_X of X as defined in Section 1. Let $\overline{R}$ be the average of scalar curvature of a Kähler metric in the Kähler class, which is independent of choice of the metric in the Kähler class. Then we prove

Theorem 0.1. *Let M be an n-dimensional polarized toric manifold and X be an extremal vector field associated to the polarized Kähler class on M. Suppose that for each $i = 1, \ldots, d$, it holds*

$$\overline{R} < \frac{n+1}{\lambda_i}, \quad \text{if } \theta_X \equiv 0, \tag{0.2}$$

or

$$\overline{R} + \|\theta_X\|_+ \leqslant \frac{n+1}{\lambda_i}, \quad \text{if } \theta_X \text{ is not zero}, \tag{0.3}$$

where $\lambda_i > 0$ are d numbers defined as in (0.1). *Then M is K-stable relatively to $\mathbb{C}^*$-action induced by X for toric degenerations.*

We will also show that the relative K-semistability is a necessary condition for the existence of extremal metrics on a polarized toric manifold for toric degenerations (Proposition 4.5). In a subsequent paper [28], the authors can improve that the relative K-stability is a necessary condition for the existence of extremal metrics.

In Section 6, several computations of $\|\theta_X\|_+$ will be given to verify conditions (0.2) or (0.3) on some Kähler classes of toric surfaces. Note that $\theta_X \equiv 0$ is equivalent to that the Futaki invariant vanishes [8]. So in case that M is a toric Fano manifold with the vanishing Futaki invariant, condition (0.2) in Theorem 0.1 is trivial on the canonical Kähler class $2\pi c_1(M)$ since $\overline{R} = n$ and all λ_i are 1. Thus Theorem 0.1 is always true for these toric manifolds. If the Futaki invariant is not zero, (0.3) becomes

$$\|\theta_X\|_+ \leqslant 1. \tag{0.4}$$

We will verify (0.4) on toric Fano surfaces. It is interesting to ask whether (0.4) is true for higher-dimensional toric Fano manifolds or not.

[2] It is no need to assume the existence of extremal metrics on M according to [8].