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Configurations in abelian categories. III. Stability conditions and identities

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Abstract

This is the third in a series on *configurations* in an abelian category $\mathcal{A}$. Given a finite poset $(I, \preccurlyeq)$, an $(I, \preccurlyeq)$-*configuration* (σ, ι, π) is a finite collection of objects $\sigma(J)$ and morphisms $\iota(J, K)$ or $\pi(J, K) : \sigma(J) \to \sigma(K)$ in $\mathcal{A}$ satisfying some axioms, where J, K are subsets of I. Configurations describe how an object X in $\mathcal{A}$ decomposes into subobjects.

The first paper defined configurations and studied moduli spaces of configurations in $\mathcal{A}$, using the theory of Artin stacks. It showed well-behaved moduli stacks $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ of objects and configurations in $\mathcal{A}$ exist when $\mathcal{A}$ is the abelian category $\mathrm{coh}(P)$ of coherent sheaves on a projective scheme P, or mod-$\mathbb{K}Q$ of representations of a quiver Q. The second studied algebras of *constructible functions* and *stack functions* on $\mathfrak{Obj}_{\mathcal{A}}$.

This paper introduces (*weak*) *stability conditions* $(\tau, T, \leqslant)$ on $\mathcal{A}$. We show the moduli spaces $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{si}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ of τ-semistable, indecomposable τ-semistable and τ-stable objects in class α are *constructible sets* in $\mathfrak{Obj}_{\mathcal{A}}$, and some associated configuration moduli spaces $\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{si}}, \mathcal{M}_{\mathrm{st}}, \mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{si}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ constructible in $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$, so their characteristic functions $\delta_{\mathrm{ss}}^{\alpha}, \delta_{\mathrm{si}}^{\alpha}, \delta_{\mathrm{st}}^{\alpha}(\tau)$ and $\delta_{\mathrm{ss}}, \ldots$, $\delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ are constructible.

We prove many identities relating these constructible functions, and their stack function analogues, under pushforwards. We introduce interesting algebras $\mathcal{H}_{\tau}^{\mathrm{pa}}, \mathcal{H}_{\tau}^{\mathrm{to}}, \overline{\mathcal{H}}_{\tau}^{\mathrm{pa}}, \overline{\mathcal{H}}_{\tau}^{\mathrm{to}}$ of constructible and stack functions, and study their structure. In the fourth paper we show $\mathcal{H}_{\tau}^{\mathrm{pa}}, \ldots, \overline{\mathcal{H}}_{\tau}^{\mathrm{to}}$ are independent of $(\tau, T, \leqslant)$, and construct *invariants* of $\mathcal{A}, (\tau, T, \leqslant)$.

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1. Introduction

This is the third in a series of papers [9–11] on *configurations*. Given an abelian category $\mathcal{A}$ and a finite partially ordered set (poset) $(I, \preccurlyeq)$, we define an $(I, \preccurlyeq)$-*configuration* (σ, ι, π) in $\mathcal{A}$ to be a collection of objects $\sigma(J)$ and morphisms $\iota(J, K)$ or $\pi(J, K): \sigma(J) \to \sigma(K)$ in $\mathcal{A}$ satisfying certain axioms, for $J, K \subseteq I$.

The first paper [9] defined configurations, developed their basic properties, and studied moduli spaces of configurations in $\mathcal{A}$, using the theory of Artin stacks. It proved well-behaved moduli stacks $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ of objects and configurations exist when $\mathcal{A}$ is the abelian category $\mathrm{coh}(P)$ of coherent sheaves on a projective $\mathbb{K}$-scheme P, or mod-$\mathbb{K}Q$ of representations of a quiver Q. The second [10] defined and studied infinite-dimensional algebras of *constructible functions* and *stack functions* on $\mathfrak{Obj}_{\mathcal{A}}$, motivated by *Ringel–Hall algebras*.

Configurations are a tool for describing how an object X in $\mathcal{A}$ decomposes into subobjects. They are especially useful for studying *stability conditions* on $\mathcal{A}$, which are the subject of this paper. Given a stability condition $(\tau, T, \leqslant)$ on $\mathcal{A}$, objects X in $\mathcal{A}$ are called τ-*semistable*, τ-*stable* or τ-*unstable* according to whether subobjects $S \subset X$ with $S \neq 0, X$ have $\tau([S]) \leqslant \tau([X])$, $\tau([S]) < \tau([X])$, or $\tau([S]) > \tau([X])$. Examples of stability conditions include slope functions, and Gieseker stability of coherent sheaves.

We also define *weak stability conditions*, which include μ-stability and purity for coherent sheaves. When $(\tau, T, \leqslant)$ is a weak stability condition each $X \in \mathcal{A}$ has a unique *Harder–Narasimhan filtration* by subobjects $0 = A_0 \subset \cdots \subset A_n = X$ whose factors $S_k = A_k/A_{k-1}$ are τ-semistable with $\tau([S_1]) > \cdots > \tau([S_n])$. If $(\tau, T, \leqslant)$ is also a stability condition each τ-semistable X has a (nonunique) filtration with (unique) τ-stable factors S_k with $\tau([S_k]) = \tau([X])$. Thus, τ-stability is well-behaved for stability conditions but badly behaved for weak stability conditions, though τ-semistability is well-behaved for both.

We form moduli spaces $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}, \mathrm{Obj}^{\alpha}_{\mathrm{si}}, \mathrm{Obj}^{\alpha}_{\mathrm{st}}(\tau)$ of τ-semistable, τ-semistable-indecomposable and τ-stable objects in class α in $K(\mathcal{A})$, and moduli spaces $\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{si}}, \mathcal{M}_{\mathrm{st}}, \mathcal{M}^{\mathrm{b}}_{\mathrm{ss}}, \mathcal{M}^{\mathrm{b}}_{\mathrm{si}}, \mathcal{M}^{\mathrm{b}}_{\mathrm{st}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ of $(I, \preccurlyeq)$-configurations (σ, ι, π) in which the smallest objects $\sigma(\{i\})$ for $i \in I$ lie in $\mathrm{Obj}^{\kappa(i)}_{\mathrm{ss}}, \mathrm{Obj}^{\kappa(i)}_{\mathrm{si}}, \mathrm{Obj}^{\kappa(i)}_{\mathrm{st}}(\tau)$, and (σ, ι, π) is *best* for $\mathcal{M}^{\mathrm{b}}_{*}(\cdots)_{\mathcal{A}}$. It is a central, and unconventional, feature of our approach that we regard these not as spaces in their own right, but as *constructible sets* in the stacks $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$, so their characteristic functions $\delta^{\alpha}_{\mathrm{ss}}, \delta^{\alpha}_{\mathrm{si}}, \delta^{\alpha}_{\mathrm{st}}(\tau)$ and $\delta_{\mathrm{ss}}, \ldots, \delta^{\mathrm{b}}_{\mathrm{st}}(I, \preccurlyeq, \kappa, \tau)$ are *constructible functions*.

This has a number of ramifications. Firstly, our approach is helpful for comparing moduli spaces, and especially for understanding how $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ changes when we vary $(\tau, T, \leqslant)$, as we are not comparing two different varieties, but two subsets of the same stack $\mathfrak{Obj}_{\mathcal{A}}$. Secondly, $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ is a set of isomorphism classes, not of S-equivalence classes. This is better for studying the family of ways a τ-semistable X may be broken into τ-stable factors. But it means $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ is *not a well-behaved topological space*, as it may not be Hausdorff, for instance. Because of this, in [11] we focus on 'motivic' invariants of constructible sets such as Euler characteristics and virtual Poincaré polynomials.

We begin in Section 2 with background on abelian categories, constructible sets and functions, and *stack functions* on Artin $\mathbb{K}$-stacks, following [7,8]. Stack functions are a universal generalization of constructible functions, containing more information. Section 3 reviews the previous