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Brill–Gordan loci, transvectants and an analogue of the Foulkes conjecture

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Abstract

The hypersurfaces of degree d in the projective space $\mathbb{P}^n$ correspond to points of $\mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$. Now assume $d = 2e$ is even, and let $X_{(n,d)} \subseteq \mathbb{P}^N$ denote the subvariety of two e-fold hyperplanes. We exhibit an upper bound on the Castelnuovo regularity of the ideal of $X_{(n,d)}$, and show that this variety is r-normal for $r \geqslant 2$. The latter result is representation-theoretic, and says that a certain GL_{n+1}-equivariant morphism

$$S_r\big(S_{2e}\big(\mathbf{C}^{n+1}\big)\big) \to S_2\big(S_{re}\big(\mathbf{C}^{n+1}\big)\big)$$

is surjective for $r \geqslant 2$; a statement which is reminiscent of the Foulkes–Howe conjecture. For its proof, we reduce the statement to the case $n = 1$, and then show that certain transvectants of binary forms are nonzero. The latter part uses explicit calculations with Feynman diagrams and hypergeometric series. For ternary quartics and binary d-ics, we give explicit generators for the defining ideal of $X_{(n,d)}$ expressed in the language of classical invariant theory.

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1. Introduction

1.1. The Foulkes–Howe conjecture

One of the major problems in the representation theory of the general linear group is understanding the composition of Schur functors, variously known as plethysm or 'external product' of symmetric functions. Even in the 'simple' case of a composition of symmetric powers $S_r(S_m(\mathbf{C}^{n+1}))$ (which is the space of homogeneous polynomials of degree r in the coefficients of a generic homogeneous polynomial of degree m in $n+1$ variables), very little is known about its decomposition into irreducible representations of SL_{n+1}. While trying to shed light on this problem, R. Howe [42] constructed a natural equivariant map

$$S_r\big(S_m\big(\mathbf{C}^{n+1}\big)\big) \to S_m\big(S_r\big(\mathbf{C}^{n+1}\big)\big).$$

He conjectured that the map is injective if $r \leqslant m$, and surjective if $r \geqslant m$, thereby giving a more precise form to a question raised by H.O. Foulkes [31]. (See [9,13,26] for recent results and further references.) More generally, for any integer $e \geqslant 1$, there is an equivariant map

$$S_r\big(S_{me}\big(\mathbf{C}^{n+1}\big)\big) \to S_m\big(S_{re}\big(\mathbf{C}^{n+1}\big)\big), \tag{1}$$

which reduces to Howe's map for $e = 1$. (An explicit definition of the map will be given in Section 5.) An immediate question is whether this more general map also is surjective when $r \geqslant m$. Our main result says that this is so for $m = 2$.

Theorem 1.1. *The map*

$$\alpha_r : S_r\big(S_{2e}\big(\mathbf{C}^{n+1}\big)\big) \to S_2\big(S_{re}\big(\mathbf{C}^{n+1}\big)\big)$$

is surjective for $r \geqslant 2$.

Remark 1.2. The following result was recently proved by Rebecca Vessenes in her thesis (see [59, Theorem 1]): *For any partition λ and $r \geqslant 2$, the multiplicity of the irreducible Schur module $S_\lambda(\mathbf{C}^{n+1})$ in $S_r(S_{2e}(\mathbf{C}^{n+1}))$ is at least equal to its multiplicity in $S_2(S_{re}(\mathbf{C}^{n+1}))$.* The theorem above of course implies this. The technique of tableaux counting used by her gives a similar (but slightly weaker) result (see [59, Theorem 2]): *For $r \geqslant 3$, any module $S_\lambda(\mathbf{C}^{n+1})$ which has positive multiplicity in $S_3(S_{re}(\mathbf{C}^{n+1}))$ also has positive multiplicity in $S_r(S_{3e}(\mathbf{C}^{n+1}))$.* This is inaccessible by our method as it stands.

Remark 1.3. To the best of our knowledge, the map (1) is first considered by Brion (see [12, §1.3]). He shows that there exists a constant $C(m, e, n)$, such that (1) is surjective for $r \geqslant C(m, e, n)$.

1.2. Brill–Gordan loci

In fact, we discovered Theorem 1.1 in the course of an entirely different line of inquiry. The context is as follows.

The set of hypersurfaces of degree d in $\mathbb{P}^n$ is parametrized by the projective space $\mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$. Assume that d is even (say $d = 2e$), and consider the subset of hypersurfaces