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An Auslander-type result for Gorenstein-projective modules ☆

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Abstract

An artin algebra A is said to be CM-finite if there are only finitely many, up to isomorphisms, indecomposable finitely generated Gorenstein-projective A-modules. We prove that for a Gorenstein artin algebra, it is CM-finite if and only if every its Gorenstein-projective module is a direct sum of finitely generated Gorenstein-projective modules. This is an analogue of Auslander's theorem on algebras of finite representation type [M. Auslander, A functorial approach to representation theory, in: Representations of Algebras, Workshop Notes of the Third Internat. Conference, in: Lecture Notes in Math., vol. 944, Springer-Verlag, Berlin, 1982, pp. 105–179; M. Auslander, Representation theory of artin algebras II, Comm. Algebra (1974) 269–310].

Keywords: Gorenstein-projective modules; Triangulated categories; Dualizing varieties

1. Introduction

Let A be an artin R-algebra, where R is a commutative artinian ring. Denote by A-mod (resp. A-mod) the category of (resp. finitely generated) left A-modules. Denote by A-Proj (resp. A-proj) the category of (resp. finitely generated) projective A-modules. Following [21], a chain complex $P^\bullet$ of projective A-modules is defined to be *totally-acyclic*, if for every projective

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module $Q \in A$-Proj the Hom-complexes $\mathrm{Hom}_A(Q, P^\bullet)$ and $\mathrm{Hom}_A(P^\bullet, Q)$ are exact. A module M is said to be *Gorenstein-projective* if there exists a totally-acyclic complex $P^\bullet$ such that the 0th cocycle $Z^0(P^\bullet) = M$. Denote by A-GProj the full subcategory of Gorenstein-projective modules. Similarly, we define finitely generated Gorenstein-projective modules by replacing all modules above by finitely generated ones, and we also get the category A-Gproj of finitely generated Gorenstein-projective modules [17]. It is known that $A\text{-Gproj} = A\text{-GProj} \cap A\text{-mod}$ [14, Lemma 3.4]. Finitely generated Gorenstein-projective modules are also referred as maximal Cohen–Macaulay modules. These modules play a central role in the theory of singularity [10–12, 14] and of relative homological algebra [9,17].

An artin algebra A is said to be *CM-finite* if there are only finitely many, up to isomorphisms, indecomposable finitely generated Gorenstein-projective modules. Recall that an artin algebra A is said to be of *finite representation type* if there are only finitely many isomorphism classes of indecomposable finitely generated modules. Clearly, finite representation type implies CM-finite. The converse is not true, in general.

Let us recall the following famous result of Auslander [3,4] (see also Ringel–Tachikawa [27, Corollary 4.4]):

Auslander's theorem. *An artin algebra A is of finite representation type if and only if every A-module is a direct sum of finitely generated modules, that is, A is left pure semisimple, see* [31].

Inspired by the theorem above, one may conjecture the following Auslander-type result for Gorenstein-projective modules: an artin algebra A is CM-finite if and only if every Gorenstein-projective A-module is a direct sum of finitely generated ones. However we can only prove this conjecture in a nice case.

Recall that an artin algebra A is said to be Gorenstein [19] if the regular module A has finite injective dimension both at the left and right sides. Our main result is

Main theorem. *Let A be a Gorenstein artin algebra. Then A is CM-finite if and only if every Gorenstein-projective A-module is a direct sum of finitely generated Gorenstein-projective modules.*

Note that our main result has a similar character to a result by Beligiannis [9, Proposition 11.23], and also note that similar concepts were introduced and then similar results and ideas were developed by Rump in a series of papers [28–30].

2. Proof of Main theorem

Before giving the proof, we recall some notions and known results.

2.1. Let A be an artin R-algebra. By a subcategory $\mathcal{X}$ of A-mod, we mean a full additive subcategory which is closed under taking direct summands. Let $M \in A$-mod. We recall from [6,7] that a *right $\mathcal{X}$-approximation* of M is a morphism $f: X \to M$ such that $X \in \mathcal{X}$ and every morphism from an object in $\mathcal{X}$ to M factors through f. The subcategory $\mathcal{X}$ is said to be *contravariantly-finite in* A-mod if each finitely generated modules has a right $\mathcal{X}$-approximation. Dually, one defines the notions of *left $\mathcal{X}$-approximations* and *covariantly-finite* subcategories. The subcategory $\mathcal{X}$ is said to be *functorially-finite in* A-mod if it is contravariantly-finite and