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Geometric representation theory for unitary groups of operator algebras

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Abstract

Geometric realizations for the restrictions of GNS representations to unitary groups of C^*-algebras are constructed. These geometric realizations use an appropriate concept of reproducing kernels on vector bundles. To build such realizations in spaces of holomorphic sections, a class of complex coadjoint orbits of the corresponding real Banach–Lie groups is described and some homogeneous holomorphic Hermitian vector bundles that are naturally associated with the coadjoint orbits are constructed.

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1. Introduction

The study of geometric properties of state spaces is a basic topic in the theory of operator algebras (see, e.g., [2] and [3]). The GNS construction produces representations of operator algebras out of states. From this point of view, we think it interesting to investigate the geometry behind these representations.

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One method to do this is to proceed as in the theory of geometric realizations of Lie group representations (see, e.g., [22,24,25,31]) and to try to build the representation spaces as spaces of sections of certain vector bundles. The basic ingredient in this construction is the reproducing kernel Hilbert space (see, e.g., [6,7,12,27,31,36,38]).

In the present paper we show that the aforementioned method can indeed be applied to the case of group representations obtained by restricting GNS representations to unitary groups of C^*-algebras. More precisely, for these representations, we construct one-to-one intertwining operators from the representation spaces onto reproducing kernel Hilbert spaces of sections of certain Hermitian vector bundles (Theorem 5.4). The construction of these vector bundles is based on a choice of a sub-C^*-algebra that is related in a suitable way to the state involved in the GNS construction (see Construction 3.1).

It turns out that, in the case of normal states of W^*-algebras, there is a natural choice of the subalgebra (namely the centralizer subalgebra), and the base of the corresponding vector bundle is just one of the symplectic leaves studied in our previous paper [11]. Since the corresponding symplectic leaves are just unitary orbits of states, the geometric representation theory initiated in the present paper provides, in particular, a geometric interpretation of the result in [26], namely the equivalence class of an irreducible GNS representation only depends on the unitary orbit of the corresponding pure state.

In [17] and references therein one can find several interesting results regarding the classification of unitary group representations of various operator algebras. The point of the present paper is to show that some of these representations (namely the ones obtained by restricting GNS representations to unitary groups) can be realized geometrically following the pattern of the classical Borel–Weil theorem for compact groups.

This raises the challenging problem of finding geometric realizations of more general representations of unitary groups of operator algebras. Similar results for other classes of infinite-dimensional groups have been already obtained: see [23,30,42] for direct limit groups, and [15,32,33] for groups related to operator ideals. The same problem of geometric realizations for representations of the restricted unitary group was raised at the end of [16].

The structure of the paper is as follows. Since the reproducing kernels we need in the present paper show up most naturally in a C^*-algebraic setting (Construction 5.1), we establish in Section 2 the appropriate versions of a number of results in [11]. Section 3 gives a general construction of homogeneous Hermitian vector bundles associated with GNS representations. Section 4 is devoted to the concept of reproducing kernel suitable for the applications we have in mind. In Section 5 we construct such reproducing kernels out of GNS representations and we prove our main theorems on geometric realizations of GNS representations (Theorems 5.4 and 5.8).

2. Coadjoint orbits and C^*-algebras with finite traces

In this section we extend to a C^*-algebraic framework a number of results that were proved in [11] for symplectic leaves in preduals of W^*-algebras. We begin by establishing some notation that will be used throughout the paper.

Notation 2.1. For a unital C^*-algebra A with the unit $\mathbf{1}$ we shall use the following notation:

$$\{a\}' = \{b \in A \mid ab = ba\} \quad \text{whenever } a \in A,$$