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Zeta functions of finite graphs and coverings, III

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Abstract

A graph theoretical analog of Brauer–Siegel theory for zeta functions of number fields is developed using the theory of Artin L-functions for Galois coverings of graphs from parts I and II. In the process, we discuss possible versions of the Riemann hypothesis for the Ihara zeta function of an irregular graph.

Keywords: Ihara zeta function; Graph theory Riemann hypothesis; Ramanujan graph; Artin L-function for Galois graph covering

1. Introduction

In our previous two papers [12,13] we developed the theory of zeta and L-functions of graphs and covering graphs. Here zeta and L-functions are reciprocals of polynomials which means these functions have poles not zeros. Just as number theorists are interested in the locations of the zeros of number theoretic zeta and L-functions, thanks to applications to the distribution of primes, we are interested in knowing the locations of the poles of graph-theoretic zeta and L-functions. We study an analog of Brauer–Siegel theory for the zeta functions of number fields (see Stark [11] or Lang [6]). As explained below, this is a necessary step in the discussion of the distribution of primes.

We will always assume that our graphs X are finite, connected, *rank* $\geqslant 1$ *with no danglers* (i.e., degree 1 vertices). Let us recall some of the definitions basic to Stark and Terras [12,13].

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If X is any connected finite undirected graph with vertex set V and (undirected) edge set E, we orient its edges arbitrarily and obtain $2|E| = 2m$ oriented edges. We always use the following *oriented edge labelling*

$$e_1, e_2, \dots, e_m, e_{m+1} = e_1^{-1}, \dots, e_{2m} = e_m^{-1}. \tag{1}$$

"*Primes*" $[C]$ in X are equivalence classes of closed backtrackless tailless primitive paths C. Write $C = a_1 a_2 \cdots a_s$, where a_j is an oriented edge of X. The *length* of C is $\nu(C) = s$. *Backtrackless* means that $a_{i+1} \neq a_i^{-1}$, for all i. *Tailless* means that $a_s \neq a_1^{-1}$. The *equivalence class* $[C]$ is the set

$$[C] = \{a_1 a_2 \cdots a_s, a_2 a_3 \cdots a_s a_1, \dots, a_s a_1 \cdots a_{s-1}\}.$$

$[C]$ is *primitive* means $C \neq D^m$, for any integer $m \geqslant 2$ and path D in X.

Here r_X will denote the *rank* of the fundamental group of X. We have $r_X - 1 = |E| - |V|$. Then r_X is the number of edges deleted from X to form a spanning tree. We will call such deleted edges "*cut*" edges, since there should be no confusion with the other meaning of cut edge.

Next let us define an *unramified finite covering* graph Y over X (*written* Y/X) in the case that the graphs have no loops or multiple edges. In this case, Y covers X means that there is a covering map $\pi : Y \to X$ such that π is an onto graph map and for each $x \in X$ and each $y \in \pi^{-1}(x)$, the set of points adjacent to y in Y is mapped by π 1-1, onto the set of points adjacent to x in X. We always consider connected coverings Y of the connected graph X obtained by viewing the d sheets of Y as copies of a spanning tree in X. If the graphs have loops and multiple edges, one must be a little more precise about the definition of covering graph. See Stark and Terras [13].

A *d-sheeted* (*unramified*) *graph covering Y/X* is *normal* iff there are d graph automorphisms $\sigma : Y \to Y$ such that $\pi\sigma(y) = \pi(y)$, for all $y \in Y$. Then *Gal*(Y/X), the *Galois group* of Y/X, is the set of all these σ's.

Recall that the *Ihara zeta function* of X is defined at $u \in \mathbb{C}$, for $|u|$ sufficiently small, by

$$\zeta_X(u) = \prod_{[C]} \bigl(1 - u^{\nu(C)}\bigr)^{-1}, \tag{2}$$

where $[C]$ runs over the primes of X.

As a power series in the complex variable u,

$$\zeta_X(u) = \sum_{n=0}^{\infty} a_n u^n, \tag{3}$$

where each coefficient $a_n \geqslant 0$. Thus, by a classic theorem of Landau, both the series (3) and the product (2) will converge absolutely in a circle $|u| < R$ with a singularity (pole of order 1 for connected X) at $u = R$.

Definition 1. R_X is the *radius of the largest circle of convergence* of the Ihara zeta function and $\omega_X = 1/R_X$.

When X is a $(q+1)$-regular graph, $R_X = 1/q$ and $\omega_X = q$. As with the Dedekind zeta function, $\zeta_X(u)$ has a meromorphic continuation to the entire complex u-plane, but now $\zeta_X(u)^{-1}$ is