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The two-phase membrane problem—An intersection-comparison approach to the regularity at branch points☆

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Abstract

For the two-phase membrane problem

$$\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}},$$

where $\lambda_+ > 0$ and $\lambda_- > 0$, we prove in two dimensions that the free boundary is in a neighborhood of each "branch point" the union of two C^1-graphs. We also obtain a stability result with respect to perturbations of the boundary data. Our analysis uses an intersection-comparison approach based on the Aleksandrov reflection.

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In higher dimensions we show that the free boundary has finite $(n-1)$-dimensional Hausdorff measure.

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1. Introduction

In this paper we study the regularity of the obstacle-problem-like equation

$$\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}} \quad \text{in } \Omega, \tag{1.1}$$

where $\lambda_+ > 0, \lambda_- > 0$ and $\Omega \subset \mathbf{R}^n$ is a given domain. Physically the equation arises for example as the "two-phase membrane problem": consider an elastic membrane touching the planar phase boundary between two liquid/gaseous phases with constant densities $\rho_1 > \rho_2$ in a gravity field, for example water and air. If the constant density ρ_m of the membrane satisfies $\rho_1 > \rho_m > \rho_2$, then the membrane is being buoyed up in the phase with higher density and pulled down in the phase with lesser density, so the equilibrium state can be described by Eq. (1.1).

Properties of the solution, etc. have been derived by Weiss in [17] and by Uraltseva in [14]. Moreover, in [13], Shahgholian et al. gave a complete characterization of global two-phase solutions satisfying a quadratic growth condition at the two-phase free boundary point 0 and at infinity. It turned out that each such solution coincides after rotation with the one-dimensional solution (Fig. 1) $u(x) = \frac{\lambda_+}{4}\max(x_n,0)^2 - \frac{\lambda_-}{4}\min(x_n,0)^2$. In particular, this implies that each blow-up limit u_0 at so-called "branch points", $\Omega \cap \partial\{u>0\} \cap \partial\{u<0\} \cap \{\nabla u = 0\}$, is after rotation of the form $u_0(x) = \frac{\lambda_+}{4}\max(x_n,0)^2 - \frac{\lambda_-}{4}\min(x_n,0)^2$. Note that the nomenclature "branch point" is abusive in the sense that it does *not necessarily* imply a bifurcation of the free boundary at that point (see Fig. 2). Also there *are* one-phase bifurcation points of the free boundary that are not included in our class of branch points. Nevertheless, it makes sense to speak of branch points because *generically* a bifurcation occurs at those points.

In this paper we prove (cf. Theorem 4.1) that *in two dimensions* the free boundary is in a neighborhood of each branch point the union of (at most) two C^1-graphs. As application we obtain the following stability result: If the free boundary contains no singular one-phase point for certain boundary data (B_0), then for boundary data (B) close to (B_0) the free boundary consists of C^1-arcs converging to those of (B) (cf. Theorem 5.1).

In higher dimensions we derive an estimate for the $(n-1)$-dimensional Hausdorff measure of the free boundary.