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Advances in Mathematics 212 (2007) 1–36
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Discrete orthogonal polynomials and difference equations of several variables

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Received 1 August 2005; accepted 25 September 2006

Available online 3 November 2006

Communicated by Michael J. Hopkins

Abstract

The goal of this work is to characterize all second order difference operators of several variables that have discrete orthogonal polynomials as eigenfunctions. Under some mild assumptions, we give a complete solution of the problem.

MSC: 42C05; 33C45

Keywords: Discrete orthogonal polynomials; Second order difference equations; Several variables

Contents

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2006.09.012

1. Introduction

The goal of this study is to characterize all second order difference equations of several variables that have discrete orthogonal polynomials as eigenfunctions. More precisely, we consider difference operators of the form

$$D = \sum_{1 \leqslant i,j \leqslant d} A_{i,j} \Delta_i \nabla_j + \sum_{i=1}^{d} B_i \Delta_i + C\mathcal{I},$$

where Δ_i and ∇_i are the forward and backward operator in the direction of the ith coordinate of $\mathbb{R}^d$, respectively, $\mathcal{I}$ is the identity operator, $A_{i,j}$, B_i and C are functions of $x \in \mathbb{R}^d$. The discrete orthogonal polynomials in our consideration are polynomials that are orthogonal with respect to an inner product of the form

$$\langle f, g \rangle = \sum_{x \in V} f(x) g(x) W(x),$$

where V is a lattice set in $\mathbb{R}^d$ and W is some positive weight function on V. There is a close correlation between D, W and V. Some restrictions need to be imposed on V due to the complexity of the geometry in higher dimensions. Under some mild and, we believe, reasonable assumptions on V, we give a complete solution of the problem.

For $d = 1$, the one-dimensional case, the classification problem was studied by several authors early in the last century, we refer to [1,6] for references. It was found that the classical discrete orthogonal polynomials, namely, Hahn polynomials, Meixner polynomials, Krawtchouk polynomials, and Charlier polynomials are eigenfunctions of second order difference operators on the real line, and these are believed to be the only ones that have such property. To our great surprise, however, another family of solutions turns up when we analyze the problem carefully. The difference equation satisfied by this family of solutions is similar to that of Hahn polynomials but the parameters need to be chosen in a different way. In other words, the difference equation has two separate families of solutions when the parameters are chosen differently.

The same phenomenon also appears in the case of two variables. The second order difference equations that have orthogonal polynomials as eigenfunctions were identified in [12], but not all families of orthogonal polynomials were found. In fact, viewing it as an analogue of second order differential operators that have orthogonal polynomials as eigenfunctions, only one family of solutions that had been known in the literature was identified for each difference equation. The solutions of difference equations, however, turn out to be far richer than that of differential equations. For example, in the case of quadratic eigenvalues, only Hahn polynomials on the set