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Generalized characters of the symmetric group

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Abstract

Normalized irreducible characters of the symmetric group $S(n)$ can be understood as zonal spherical functions of the Gelfand pair $(S(n) \times S(n), \operatorname{diag} S(n))$. They form an orthogonal basis in the space of the functions on the group $S(n)$ invariant with respect to conjugations by $S(n)$. In this paper we consider a different Gelfand pair connected with the symmetric group, that is an "unbalanced" Gelfand pair $(S(n) \times S(n-1), \operatorname{diag} S(n-1))$. Zonal spherical functions of this Gelfand pair form an orthogonal basis in a larger space of functions on $S(n)$, namely in the space of functions invariant with respect to conjugations by $S(n-1)$. We refer to these zonal spherical functions as normalized *generalized* characters of $S(n)$. The main discovery of the present paper is that these generalized characters can be computed on the same level as the irreducible characters of the symmetric group. The paper gives a Murnaghan–Nakayama type rule, a Frobenius type formula, and an analogue of the determinantal formula for the generalized characters of $S(n)$.

Keywords: Symmetric group; Gelfand pairs; Characters; Symmetric functions

1. Introduction

1.1. Preliminaries and formulation of the problem

One of the central goals of the representation theory of finite groups is in computation of characters of irreducible representations. When a group under considerations is the symmetric group, $S(n)$, the irreducible characters can be computed using either the Frobenius formula, or

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the determinantal formula, or the Murnaghan–Nakayama rule (see, for example, Macdonald [10], Sagan [21], Stanley [22]).

Let Λ denote the algebra of symmetric functions, which is a graded algebra, isomorphic to the algebra of polynomials in the power sums $p_1, p_2, \ldots$. If we define $p_\rho = p_{\rho_1} p_{\rho_2} \ldots$ for each partition $\rho = (\rho_1, \rho_2, \ldots)$, then the p_ρ form a homogeneous basis in Λ. (As in Macdonald [10] we identify each partition with its Young diagram.) Another natural homogeneous basis in Λ is formed by the Schur functions s_λ indexed by Young diagrams λ. The Frobenius formula is

$$p_\rho = \sum_{\lambda \vdash n} \chi_\rho^\lambda s_\lambda,$$

where χ_ρ^λ is the value of the irreducible character χ^λ of the symmetric group $S(n)$ on the conjugacy class in $S(n)$ indexed by the partition ρ of n. This formula is the key result in the classical theory of characters of the symmetric group $S(n)$. It shows that the character table is the transition matrix between two bases p_ρ and s_λ in the algebra of symmetric functions Λ. The Frobenius formula follows from the fact that the Schur functions s_λ are images of χ^λ in Λ under a certain map. This map is called the characteristic map, see [10, I, §7]. Thus, if we denote this map by ch, we have

$$s_\lambda = \mathrm{ch}\bigl(\chi^\lambda\bigr).$$

Another available result on irreducible characters of $S(n)$ is the formula which represents an irreducible character, χ^λ, as an alternating sum of the induced characters (i.e. the determinantal formula). Namely, denote by η_k the identity character of $S(k)$. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is any partition of n, let η_λ denote $\eta_{\lambda_1} \cdot \eta_{\lambda_2} \cdot \cdots$. Here the multiplication, $f \cdot g$, between two characters, f and g, of, say, groups $S(k)$ and $S(m)$ is defined by the induction

$$f \cdot g = \mathrm{ind}_{S(k)\times S(m)}^{S(k+m)} (f \times g).$$

With the above notation the irreducible character χ^λ is given by

$$\chi^\lambda = \det(\eta_{\lambda_i - i + j})_{1 \leqslant i, j \leqslant n}.$$

Since $\mathrm{ch}(\eta_\lambda) = h_\lambda$, where $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$, and h_r is the rth complete symmetric function, the determinantal formula for irreducible characters is equivalent to the Jacobi–Trudi formula for the Schur symmetric functions,

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leqslant i, j \leqslant n}.$$

The Murnaghan–Nakayama rule is a recursive method to compute the irreducible characters of the symmetric groups. It can be formulated as follows. Let us say that a skew Young diagram is a border strip if it is connected and does not contain any 2×2 block of boxes. Suppose that $\pi\sigma$ is an element of the symmetric group $S(n)$, where σ is a cycle of length j, and π is a permutation of the remaining $n - j$ numbers of cycle-type ρ, ρ is a partition of $n - j$. The Murnaghan–