Arab Journal of
Mathematical Sciences

Arab J Math Sci 22 (2016) 191–206

A parameter uniform numerical method for singularly perturbed delay problems with discontinuous convection coefficient

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Received 11 March 2015; received in revised form 29 June 2015; accepted 15 July 2015
Available online 6 August 2015

Abstract. In this paper a standard numerical method with piecewise linear interpolation on Shishkin mesh is suggested to solve singularly perturbed boundary value problem for second order ordinary delay differential equations with discontinuous convection coefficient and source term. An error estimate is derived by using the supremum norm and it is of almost first order convergence. Numerical results are provided to illustrate the theoretical results.

Keywords: Singularly perturbed problem; Convection–diffusion problem; Discontinuous convection coefficient; Shishkin mesh; Delay

2010 Mathematics Subject Classification: 65L10; 65L11; 65L12

1. Introduction

Singularly perturbed ordinary differential equations with a delay are ordinary differential equations in which the highest derivative is multiplied by a small parameter and involving at least one delay term. Such type of equations arises frequently from the mathematical modelling of various practical phenomena, for example, in the modelling of the human pupil-light reflex [14], the study of bistable devices [4] and variational problems in control theory [10], etc. It is important to develop suitable numerical methods to solve singularly perturbed differential equations with a delay, whose accuracy does not depend on the parameter ε, that is the methods are uniformly convergent with respect to the parameter.

In the past, only very few people had worked in the area Numerical Methods to Singularly Perturbed Delay Differential Equation(SPDDE). But in the recent years, there has been

E-mail address: suburayan123@gmail.com.
Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2015.07.001

growing interest in this area. The authors of [12,6,16,1,2] suggested some numerical methods for singularly perturbed delay differential equations with continuous data. Recently few authors in [20,21,17] suggested some numerical method for singularly perturbed delay differential equations with discontinuous data.

In the present paper, as mentioned in the above abstract, motivated by the works of [7,3,13], we consider the following singularly perturbed boundary value problem (2.1) for second order ordinary delay differential equations with discontinuous convection coefficient and suggest a parameter uniform numerical method. It is proved that this method is uniformly convergent of order $O(N^{-1}\ln^2 N)$.

The present paper is organized as follows. In Section 2, the problem of study with discontinuous data is stated. Existence of the solution to the problem is established in Section 3. A maximum principle of the DDE is established in Section 4. Further a stability result is derived. Analytical results of the problem are derived in Section 5. The present numerical method is described in Section 6 and an error estimate is derived in Section 7. Section 8 presents numerical results.

2. Statement of the problem

Through out the paper, C, C_1 denote generic positive constants independent of the singular perturbation parameter ε and the discretization parameter N of the discrete problem. Further, I_N denotes $\{0,1,\dots,N\}$. The supremum norm is used for studying the convergence of the numerical solution to the exact solution to a singular perturbation problem: $\|u\|_{\Omega} = \sup_{x\in\Omega}|u(x)|$.

Motivated by the works of [8,3,13], we consider the following BVP for SPDDE.

Find $u \in Y = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$ such that

$$\begin{cases} -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x-1) = f(x), & x \in \Omega^*, \\ u(x) = \phi(x), \quad x \in [-1,0], & u(2) = l, \end{cases} \tag{2.1}$$

$$a(x) = \begin{cases} a_1(x), & x \in [0,1], \\ a_2(x), & x \in (1,2], \end{cases} \qquad f(x) = \begin{cases} f_1(x), & x \in [0,1], \\ f_2(x), & x \in (1,2], \end{cases}$$

$$a_1(1-) \neq a_2(1+), \qquad f_1(1-) \neq f_2(1+),$$

$$a_1(x) \geq \alpha_1 > \alpha > 0, \qquad a_2(x) \leq -\alpha_2 < -\alpha < 0,$$

$$\alpha < \min\{\alpha_1, \alpha_2\}, \qquad \beta_0 \leq b(x) \leq \beta_1 < 0, \qquad \alpha + 2\beta_0 \geq \eta_0 > 0$$

where $0 < \varepsilon \ll 1$, a, f are sufficiently smooth and bounded in Ω^*. The function b is a sufficiently smooth function on $\overline{\Omega}$, $\Omega = (0,2)$, $\overline{\Omega} = [0,2]$, $\Omega^* = \Omega^- \cup \Omega^+$, $\Omega^- = (0,1)$, $\Omega^+ = (1,2)$ and ϕ is smooth on $[-1,0]$.

The above problem (2.1) is equivalent to

$$Pu(x) := \begin{cases} -\varepsilon u''(x) + a_1(x)u'(x) = f_1(x) - b(x)\phi(x-1), & x \in \Omega^-, \\ -\varepsilon u''(x) + a_2(x)u'(x) + b(x)u(x-1) = f_2(x), & x \in \Omega^+, \end{cases} \tag{2.2}$$

$$u(0) = \phi(0), \qquad u(1-) = u(1+), \qquad u'(1-) = u'(1+), \qquad u(2) = l,$$

where $u(1-)$ and $u(1+)$ denote the left and right limits of u at $x = 1$, respectively.