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On the positive weak almost limited operators

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Abstract. Using the concept of approximately order bounded sets with respect to a lattice seminorm, we establish some new characterizations of positive weak almost limited operators on Banach lattices. Consequently, we derive some results about the weak Dunford–Pettis* and the Dunford–Pettis* property of σ-Dedekind complete Banach lattices.

Keywords: Weak almost limited operator; The weak Dunford–Pettis* property; Banach lattice

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1. Introduction and notations

Throughout this paper X, Y will denote real Banach spaces, and E, F will denote real Banach lattices. E^+ denotes the positive cone of E and $sol\,(A)$ denotes the solid hull of a subset A of a Banach lattice. The notation $x_n \perp x_m$ will mean that the sequence (x_n) of a Banach lattice is disjoint, that is, $|x_n| \wedge |x_m| = 0$, $n \neq m$. An operator $T : E \to F$ is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. A lattice seminorm ϱ on a Banach lattice E is a seminorm such that for every $x, y \in E$, $|x| \leq |y|$ implies $\varrho\,(x) \leq \varrho\,(y)$. The closed unit ball associated to a lattice seminorm ϱ is defined by $B_\varrho = \{x \in E : \varrho(x) \leq 1\}$. The lattice operations in a Banach lattice E (resp. E') are weakly (resp. weak*) sequentially continuous if for every weakly null sequence (x_n) in E (resp. weak* null sequence (f_n) in E'), $|x_n| \to 0$ for $\sigma(E, E')$ (resp. $|f_n| \to 0$ for $\sigma(E', E)$). Finally, we will use the term

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operator $T : E \to F$ between two Banach lattices to mean a bounded linear mapping. We refer to [1,6] for unexplained terminology of Banach lattice theory and positive operators.

Several types of the Dunford–Pettis property are considered in the theory of Banach lattices. Namely, a Banach lattice E has

- the *Dunford–Pettis property*, whenever $x_n \xrightarrow{w} 0$ in E and $f_n \xrightarrow{w} 0$ in E' imply $f_n(x_n) \to 0$.
- the *Dunford–Pettis* property*, whenever $x_n \xrightarrow{w} 0$ in E and $f_n \xrightarrow{w^*} 0$ in E' imply $f_n(x_n) \to 0$.
- the *weak Dunford–Pettis property* (abb. wDP property) [7], whenever
$$x_n \perp x_m,\ x_n \xrightarrow{w} 0 \text{ in } E \text{ and } f_n \xrightarrow{w} 0 \text{ in } E' \text{ imply } f_n(x_n) \to 0.$$
- the *weak Dunford–Pettis** property (abb. wDP* property), whenever
$$x_n \xrightarrow{w} 0 \text{ in } E \text{ and } f_n \perp f_m, f_n \xrightarrow{w^*} 0 \text{ in } E' \text{ imply } f_n(x_n) \to 0.$$

The wDP* property, introduced recently by J. X. Chen et al. [3], is a weak version of the Dunford–Pettis* property and stronger than the wDP property. Note that the weak Dunford–Pettis* property is related to the so called *weak almost limited* operators. An operator $T : E \to F$ between Banach lattices is said to be weak almost limited [4], whenever

$$x_n \xrightarrow{w} 0 \text{ in } E \text{ and } f_n \perp f_m, f_n \xrightarrow{w^*} 0 \text{ in } F' \text{ imply } f_n(T(x_n)) \to 0.$$

Clearly, a Banach lattice E has the weak Dunford–Pettis* property if and only if the identity operator on E is weak almost limited.

Let us recall that an operator $T : X \to Y$ is said to be *limited* if $\|T^*(f_n)\| \to 0$ for every weak* null sequence $(f_n) \subset Y^*$. Furthermore, An operator $T : X \to E$ from a Banach space into a Banach lattice is said to be *almost limited* [5], if $\|T^*(f_n)\| \to 0$ for every disjoint weak* null sequence $(f_n) \subset E^*$. Accordingly, a Banach lattice E is said to have the *Schur property* (resp. *dual Schur property* [5]), if weakly null sequences in E are norm null (resp. disjoint weak* null sequences in E' are norm null). For a σ-Dedekind complete Banach lattice E (see [5, Theorem 3.3]), the dual Schur property coincide with the so called *dual positive Schur property* [2], that is, weak* null sequences in $(E')^+$ are norm null. Clearly, a Banach lattice E has the dual Schur property if and only if the identity operator on E is almost limited. For an operator $T : E \to F$ between Banach lattices the following implications are clear:

$$T \text{ is limited} \Rightarrow T \text{ is almost limited} \Rightarrow T \text{ is weak almost limited.}$$

However, there is a weak almost limited operator which needs not to be almost limited (and hence limited). Indeed, the identity operator $I : \ell^1 \to \ell^1$ is weak almost limited as ℓ^1 has the Schur (wDP*) property. But, as ℓ^1 does not have the dual positive Schur property [8, Proposition 2.1], $I : \ell^1 \to \ell^1$ is not almost limited. On the other hand, the identity operator on the Banach lattice c is not weak almost limited. Indeed, let $f_n \in c^* = \ell^1$ be such that $f_n = (0, \ldots, 0, 1_{(2n)}, -1_{(2n+1)}, 0, \ldots)$. Then (f_n) is a disjoint weak* null sequence in c^* [3, Example 2.1(2)], and clearly, the sequence (x_n) defined by $x_n = (0, \ldots, 0, 1_{(2n)}, 0, \ldots) \in c$ is weakly null, but $f_n(x_n) = 1$ for all n.

In this paper, using the concept of approximately order bounded sets with respect to a lattice seminorm, we establish a characterization of positive weak almost limited operators