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Positive solutions of multi-point boundary value problem of fractional differential equation☆

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Abstract. By means of two fixed-point theorems on a cone in Banach spaces, some existence and multiplicity results of positive solutions of a nonlinear fractional differential equation boundary value problem are obtained. The proofs are based upon some properties of Green's function, which are also the key of the paper.

Keywords: Fractional differential equation; Positive solution; Fixed-point theorem; Green's function

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1. Introduction

The purpose of this paper is to consider the existence and multiplicity of positive solutions of the nonlinear fractional differential equation boundary value problem (BVP for short):

$$\begin{cases} D_{0+}^{\alpha} u(t) = a(t)f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u(1) = \displaystyle\sum_{i=1}^{m} \beta_i u(\xi_i), \end{cases} \tag{1}$$

where D_{0+}^{α} is the Riemann–Liouville differential operator of order $2 < \alpha \leq 3$ and $m \geq 1$ is integer and $\xi_i, \beta_i > 0, f(\cdot,\cdot), a(\cdot)$ satisfying

(H1) $\beta_i > 0$ for $1 \leq i \leq m$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$ and $\sum_{i=1}^{m} \beta_i \xi_i^{\alpha-1} < 1$;

(H2) $a(t) \in L[0,1]$ is non-negative and not identically zero on any compact subset of $(0,1)$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

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Fractional calculus arises in many mathematical models in engineering and scientific disciplines. In fact, fractional-order models are more accurate than integer-order models in physics, mechanics, chemistry, aerodynamics, etc., see [3,6,7,5]. During the last few decades, many papers and books on fractional calculus are devoted to the solvability of initial fractional differential equations, see [12,1,9]. In recent years, many researchers focused on the solutions, especially the positive solutions of fractional differential equation boundary value problems, we refer to [4,2] and their references.

Very recently, the following BVP

$$\begin{cases} D_{0+}^{\alpha}u(t) = a(t)f(t,u(t),u'(t)), & 0<t<1, \\ u(0)=u'(0)=0, \qquad u(1)-\displaystyle\sum_{i=1}^{m}\beta_i u(\xi_i)=\lambda, \end{cases} \tag{2}$$

where $2<\alpha\le 3$, has been studied in [11]. By employing the Leggett–Williams fixed-point theorem, the author in [11] obtained the existence of three positive solutions for BVP (2). He proved the following conclusion (a key lemma, which is about some properties of Green's function $G(t,s)$ corresponding to BVP (2) and these properties are critical in employing the Leggett–Williams fixed-point theorem).

Conclusion (*See [11], Lemma 5*)**.** $G(t,s)$ satisfies the following conditions:

(i) $G(t,s)\ge 0$, $G(t,s)\le G(s,s)$ for all $s,t\in[0,1]$;

(ii) there exists a positive function $g\in C(0,1)$ such that $\min_{\gamma\le t\le\delta} G(t,s)\ge g(s)G(s,s)$, $s\in(0,1)$, where $0<\gamma<\delta<1$ and

$$g(s)=\begin{cases} \dfrac{\delta^{\alpha-1}(1-s)^{\alpha-1}-(\delta-s)^{\alpha-1}}{s^{\alpha-1}(1-s)^{\alpha-1}}, & s\in(0,m_1], \\ \left(\dfrac{\gamma}{\delta}\right)^{\alpha-1}, & s\in[m_1,1), \end{cases}$$

where $\gamma<m_1<\delta$;

(iii) $\max_{0\le t\le 1}\int_0^1 G(t,s)ds=\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}$.

We show that $G(t,s)$ mentioned above is

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\le t\le s\le 1, \end{cases} \tag{3}$$

where $2<\alpha\le 3$.

In the proof of Lemma 5 in [11], the author concludes that for $2<\alpha\le 3$,

$$\Gamma(\alpha)G(t,s)=t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}$$

is decreasing with respect to t for $t\ge s$. But, we declare that the conclusion is wrong because if we choose $\alpha=3$, $s=\frac{1}{2}$, then for $\frac{1}{2}\le t\le 1$, it is obvious that

$$\Gamma(\alpha)G(t,s)=t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}=\frac{1}{4}(-3t^2+4t-1)$$

is increasing in $[\frac{1}{2},\frac{2}{3}]$ and decreasing in $[\frac{2}{3},1]$. Thus (i) cannot be obtained, hence (ii) and (iii) are all invalid since their proofs are based upon (i). We refer to [11] for more details. In fact, the conclusions above are definitely wrong.