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Periodic solutions for a Cauchy problem on time scales

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Abstract. This paper firstly shows that there does not exist a nonzero periodic solution for a nonhomogeneous Cauchy problem by using the Laplace transformation on time scales. Secondly, two new Gronwall inequalities, which play an important role in the qualitative analysis of differential and integral equations, are established. Thirdly, by employing the contraction mapping principle, existence and uniqueness results of weighted S-asymptotically ω-periodic solutions for nonlinear Cauchy problem on time scales are obtained in an asymptotically periodic function space. Finally, some examples are presented to illustrate some of the results described here.

Keywords: Asymptotically periodic solutions; Cauchy problem; Time scales; Gronwall inequality

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1. Introduction

In 1990, the theory of dynamic equations on time scales was introduced by Stefan Hilger [9] in order to unify continuous and discrete calculus. This theory not only brought equations leading to new applications but also allowed one to get a better understanding of the subtle differences between discrete and continuous systems. One can find in Bohner and Peterson's books [2,3] most of the material needed to read this paper.

The question of existence and uniqueness of periodic solutions for differential and difference equations has attracted much attention during the last decades, because periodic motion is a very important and special phenomenon in both the natural and social sciences such as climate, food supplies, sustainable development and insect population [13,19,4,20,12].

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Meanwhile, periodic behavior of solutions for Cauchy problems arises from many fields of applied science involving celestial mechanics, biology and finance [5,6,17].

The literature concerning S-asymptotically ω-periodic solutions, as a very recent result, has been paid increasing attention [6,17,14]. Consequently, various methods have been developed for the study of the existence of periodic solutions, S-asymptotically ω-periodic solutions and weighted S-asymptotically ω-periodic solutions.

Most of the discussions in the earlier literature are devoted to the existence of periodic solutions for either differential or difference equations. In recent years, as time scales theory became established, attention has been given to study the periodic solutions on time scales [19,8,11,15,18,10,1,16]. However, to the best of the authors' knowledge, there are few papers concerning the S-asymptotically ω-periodic solutions and weighted S-asymptotically ω-periodic solutions on time scales.

Motivated by the above works, in this paper, we firstly consider the nonexistence of nonzero periodic solutions for the following nonhomogeneous Cauchy problem on time scales

$$\begin{cases} u^{\Delta}(t) = pu(t) + q(t), & t \in \mathbb{T}_0^k, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\mathbb{T}_0$ is a periodic time scale, $p \neq 0$ is a regressive constant number, $q \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$.

Secondly, we investigate the existence of S-asymptotically ω-periodic solutions and weighted S-asymptotically ω-periodic solutions for the following nonlinear differential equation

$$\begin{cases} u^{\Delta}(t) = r(t)u(t) + f(t, u), & t \in \mathbb{T}_0^k, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where $r \in \mathcal{R}^+$ is regressive, $f(t, u) \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R})$.

Wang, Fečkan and Zhou in [17] have studied fractional differential Cauchy problems with order $\alpha \in (0, 1)$. Many nice and interesting results were obtained.

The organization of this paper is as follows. In Section 2, some advanced topics in the theory of dynamic equations on time scales are recalled, meanwhile, the definitions of periodic time scales and asymptotically periodic functions are given. In Section 3, we first find the Laplace transform of a function, and prove a theorem which is exclusively for the periodic functions. Then, the nonexistence nonzero periodic solution for the problem (1.1) is studied. In Section 4, two Gronwall inequalities on time scales are obtained, existence and uniqueness results of weighted S-asymptotically ω-periodic solutions for the problem (1.2) are researched. Finally, two examples are presented to illustrate some of the results described here.

2. Preliminaries

We suppose that the reader is familiar with the basic concepts and calculus on time scales for dynamic equations. So, in this section, we first recall some advanced topics in the theory of dynamic equations on time scales which are used in what follows. For more details one can see references [2,3].

Let $\mathbb{T}$ be a time scale. A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0\}$.