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Generalized derivations as homomorphisms or anti-homomorphisms on Lie ideals

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Abstract. Let R be a prime ring of $char(R) \neq 2$, Z the center of R, and L a nonzero Lie ideal of R. If R admits a generalized derivation F associated with a derivation d which acts as a homomorphism or as anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z$. This result generalizes a theorem of Wang and You.

Keywords: Generalized derivations; Martindale ring of quotients; Prime ring; Lie ideal; Homomorphisms and anti-homomorphisms

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1. Introduction

Throughout this paper, unless specifically stated, R will be an associative ring, Z the center of R, Q its two-sided Martindale quotient ring and U its right Utumi quotient ring (some times, as in [2], U is called the maximal right ring of quotients). The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [2], for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. Recall that a ring R is prime if $xRy = 0$ implies either $x = 0$ or $y = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$, if $d(x) = [q, x]$ holds for all $x \in R$. By a generalized inner derivation on R, one usually means an additive mapping $F : R \to R$ if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that $F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y)$,

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where I_b is an inner derivation determined by b. This observation leads to the definition given in [5]: an additive mapping $F : R \to R$ is called generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Obviously any derivation is a generalized derivation. Other basic examples of generalized derivations are the following: (i) $F(x) = ax + xb$ for $a, b \in R$; (ii) $F(x) = ax$ for some $a \in R$. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$ is a generalized derivation, where c is a fixed element of R and d is a derivation of R. In [16], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation on U, and thus all generalized derivations of R will be implicitly assumed to be defined on dense right ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U (see Theorem 3, in [16]).

In [3, Theorem 3], Bell and Kappe proved that if d is a derivation of a prime ring R which acts as homomorphisms or anti-homomorphisms on a nonzero right ideal of R then $d = 0$ on R. Further Asma et al. [1], extend this result to Lie ideals of 2-torsion free prime rings. More precisely they prove that if L is a noncentral Lie ideal of R such that $u^2 \in L$, for all $u \in L$ and d acts as a homomorphism or anti-homomorphism on L, then $d = 0$. In 2007 Wang and You [19], eliminate the hypothesis $u^2 \in L$, for all $u \in L$ and prove the same result as Asma et al. [1]. To be more specific, the statement of Wang and You theorem is the following:

Theorem 1.1 (*[19, Theorem 1.2]*). *Let R be a 2-torsion free prime ring and L a nonzero Lie ideal of R. If d is a derivation of R which acts as a homomorphism or an anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z$.*

In [18], First author studies the case when the derivation d is replaced by a generalized derivation F and obtain the following: if R is a 2-torsion free prime ring and F acts as a homomorphism or an anti-homomorphism on a nonzero ideal of R, then R must be commutative. For more details related results we refer the reader to [7,8,10,11]. Our work is then motivated by the previous results. The aim of the present paper is to generalize Theorem 1.1, for generalized derivation F by using the same technique as Wang and You [19] with necessary variations.

Explicitly we shall prove the following theorem.

Theorem 1.2. *Let R be a prime ring with char$(R) \neq 2$, L a nonzero Lie ideal of R. If R admits a generalized derivation F associated with a derivation d which acts as a homomorphism or an anti-homomorphism on L, then $d = 0$ or $L \subseteq Z$.*

2. Main result

We will make frequent use of the following result due to Kharchenko [14] (see also [15]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two sided ideal of R. Let $f(x_1, \ldots, x_n, d(x_1, \ldots, x_n))$ be a differential identity in I, that is

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0 \quad \text{for all } r_1, \ldots, r_n \in I.$$

Then one of the following holds: