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Little Hankel operators on the Bergman space

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Abstract. In this paper we obtain a characterization of little Hankel operators defined on the Bergman space of the unit disk and then extend the result to vector valued Bergman spaces. We then derive from it certain asymptotic properties of little Hankel operators.

Keywords: Little Hankel operators; Toeplitz operators; Inner functions; Bergman space; Hardy space

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1. INTRODUCTION

Let $dA(z)$ be the Lebesgue area measure on the open unit disk $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ equals 1. The Bergman space $A^2(\mathbb{D})$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $K_z \in A^2(\mathbb{D})$ such that $f(z) = \langle f, K_z \rangle$ for every $f \in A^2(\mathbb{D})$. The normalized reproducing kernel k_z is the function $\frac{K_z}{\|K_z\|_2}$. Here the norm $\| \cdot \|_2$ and the inner product $\langle , \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. Then $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis [23] for $A^2(\mathbb{D})$. Let $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\overline{w})^2} = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$. Let $L^\infty(\mathbb{D})$ be the space of all essentially bounded Lebesgue measurable functions on $\mathbb{D}$ with the norm $\|f\|_\infty = ess \sup_{z \in \mathbb{D}} |f(z)|$. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $A^2(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$. The Hankel operator $H_\phi : A^2(\mathbb{D}) \to (A^2(\mathbb{D}))^\perp$ with symbol $\phi \in L^\infty(\mathbb{D})$ is defined by $H_\phi f = (I - P)(\phi f)$. The little Hankel

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operator $S_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ is defined by $S_\phi f = PJ(\phi f)$ where $J : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$ is defined as $Jf(z) = f(\overline{z})$. There are also many equivalent ways of defining little Hankel operators on the Bergman space. For example, for $\phi \in L^\infty(\mathbb{D})$, define $\Gamma_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ as $\Gamma_\phi f = P(\phi J f)$. It is not difficult to see that $\Gamma_\phi = S_{J\phi}$. Here we refer both the operators S_ϕ and Γ_ϕ as little Hankel operators on the Bergman space. Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that

(i) $(\phi_a \, o \, \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a(z)} = |k_a(z)|^2 = \frac{\left(1-|a|^2\right)^2}{|1-\overline{a}z|^4}$. Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_a f$ on $\mathbb{D}$ by $U_a f(z) = k_a(z) f(\phi_a(z))$. Notice that U_a is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$ for all $a \in \mathbb{D}$. Further, it can be verified that $U_a^2 = I$, the identity operator, $U_a^* = U_a, U_a(A^2(\mathbb{D})) \subset A^2(\mathbb{D})$ and $U_a((A^2(\mathbb{D}))^\perp) \subset (A^2(\mathbb{D}))^\perp$ for all $a \in \mathbb{D}$. Thus $U_a P = P U_a$ for all $a \in \mathbb{D}$.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $L^2(\mathbb{T})$ be the space of square integrable, measurable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure on $\mathbb{T}$. The sequence $\{\varepsilon_n(z)\}_{n=-\infty}^\infty = \{e^{in\theta}\}_{n=-\infty}^\infty$ forms an orthonormal basis for $L^2(\mathbb{T})$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are $c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta, n \in \mathbb{Z},$ where $\mathbb{Z}$ is the set of all integers. The Hardy space $H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ consisting of functions f with $c_n(f) = 0$ for all negative integers n. Since $c_n = c_n(.)$ is a bounded linear functional on $L^2(\mathbb{T})$ for any fixed n, and $H^2(\mathbb{T}) = \bigcap_{n<0} \ker \ c_n$, it follows that $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$ and therefore a Hilbert space.

Let $L^\infty(\mathbb{T})$ be the space of all essentially bounded measurable functions on $\mathbb{T}$. For $\phi \in L^\infty(\mathbb{T})$, we define the multiplication operator M_ϕ from $L^2(\mathbb{T})$ into itself by $M_\phi f = \phi f$. The multiplication here is to be understood in the obvious sense, namely it is the pointwise one: $(\phi f)(e^{i\theta}) = \phi(e^{i\theta}) f(e^{i\theta})$ for all $\theta \in [0, 2\pi]$. Let $\mathbb{P}$ be the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. For $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator $\mathcal{T}_\phi$ on $H^2(\mathbb{T})$ is defined by $\mathcal{T}_\phi f = \mathbb{P}(\phi f)$ for f in $H^2(\mathbb{T})$ and the Hankel operator $\mathbb{S}_\phi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ as $\mathbb{S}_\phi f = \mathbb{P}\mathbb{J}(\phi f)$ where $\mathbb{J}(e^{i\theta}) = e^{-i\theta}, 0 \leq \theta \leq 2\pi$. The function ϕ is called the symbol of the Hankel operator $\mathbb{S}_\phi$. Let $H^2(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$ which are harmonic extensions of functions in $H^2(\mathbb{T})$. It is not very important [10] to distinguish $H^2(\mathbb{D})$ from $H^2(\mathbb{T})$. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself and $\mathcal{LC}(H)$ be the set of all compact operators in $\mathcal{L}(H)$.

In 1964, Brown and Halmos [5] established that $T \in \mathcal{L}(H^2(\mathbb{T}))$ is a Toeplitz operator if and only if $\mathcal{T}_z^* T \mathcal{T}_z = T$. Further, they have shown that $T\mathcal{T}_z = \mathcal{T}_z T$ if and only if $T = \mathcal{T}_\phi, \phi \in H^\infty(\mathbb{T})$. In 1996, Cao [6] established that there is no nonzero bounded operator $T \in \mathcal{L}(A^2(\mathbb{D}))$ such that $T_z^* T T_z = T$. Englis [11] in 1988 proved that if $A, B \in \mathcal{L}(A^2(\mathbb{D}))$ are such that $AT_f B = T_f$ for all $f \in L^\infty(\mathbb{D})$ then both A and B are scalar multiples of the identity. Nehari [22] in 1957 gave a characterization of Hankel operators on the Hardy space. Nehari showed that if $H \in \mathcal{L}(H^2(\mathbb{T}))$ then $\mathcal{T}_z^* H = H\mathcal{T}_z$ if and only if there exists $\phi \in L^\infty(\mathbb{T})$ such that $H = \mathbb{S}_\phi$, a Hankel operator with symbol ϕ. In 1991, Faour [15]