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Characterization of self-adjoint domains for differential operators with interior singular points

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Abstract. We characterize the self-adjoint domains of general even order linear ordinary differential operators which have finite interior singular points in terms of real-parameter solutions of the differential equation. For the purpose we constructed a direct sum space. By the theory of direct sum space and the decomposition of the corresponding maximal domain, we give this complete and analytic characterization in terms of limit-circle solutions. This is for endpoints which are regular or singular and for arbitrary deficiency index.

Keywords: Differential operators; Interior singular points; Deficiency index; Self-adjoint domains; Real-parameter solutions

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1. Introduction

W.N. Everitt and A. Zettl in [3] developed a theory of self-adjoint realizations of Sturm–Liouville problems on two intervals in the direct sum of Hilbert spaces associated with these intervals, for solving the Sturm–Liouville eigenvalue problems with interior singular points. In 1988, A.M. Krall and A. Zettl in [7] generalized the method given by Coddington [1], which obtains the characterization of self-adjoint domains by describing the boundary conditions of the domain of a conjugate differential operator, and obtains the characterization of self-adjoint domains for Sturm–Liouville differential operators with interior singular points.

As noted in [3], a simple way of getting self-adjoint operators in a direct sum Hilbert space is to take the direct sum of self-adjoint operators from each of the separate Hilbert spaces. However, there are many self-adjoint operators which are not merely the sum of self-adjoint

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operators from each of the separate intervals. These "new" self-adjoint operators involve interactions between the two intervals. Therefore in [3] the authors develop a "two-interval" theory. In particular, they characterized self-adjoint extensions of the minimal operator in the direct sum space in terms of boundary conditions. This theory was extended in [4] to higher order equations and any number of intervals, finite or infinite.

As in the case with no interior singular point the GKN characterization (see [2]) depends on maximal domain vectors. These vectors depend on the coefficients of each differential equation and this dependence is implicit and complicated. In [10] Wang, Sun and Zettl give an explicit characterization of all self-adjoint domains for singular problems in terms of the LC solutions for real λ when one endpoint a is regular and the other b is singular. Under the assumption that the differential equation $My = \lambda wy$ has d linearly independent solutions in H for some real λ, for $m = 2d-2k$, they constructed solutions $u_1, \dots, u_m$ and $u_{m+1}, \dots, u_d$ of the equation all lying in H such that the solutions u_j for $j > m$ do not contribute to the boundary conditions at the singular endpoint b and the solutions $u_1, \dots, u_m$ do contribute. Thus, in analogy with the celebrated Weyl limit-point (LP) and limit-circle (LC) cases for second order i.e. Sturm–Liouville problems, we say that the solutions $u_1, \dots, u_m$ are of LC type at b and $u_{m+1}, \dots, u_d$ are of LP type at b. Following [10], Hao, Wang, Sun and Zettl give a new characterization by dividing (a, b) into two intervals (a, c) and (c, b) for some $c \in (a, b)$ and using the LC solutions on each interval constructed in [10] when a and b are singular in [6]. In [9], Suo and Wang extend the characterization in [6] to two-interval case when one or two or three or four endpoints of two interval $(a_1, b_1) \cup (a_2, b_2)$ are regular and illustrate the interactions between the regular points and singular points with some examples.

In this paper we extend the characterization in [9] to the case when the differential operators which have finite interior singular points. For the purpose we firstly construct a direct sum space $H = \sum_{r=1}^{q} \oplus L^2((a_r, b_r), w_r)$ and give the corresponding notations and basic facts for direct sum space differential operators. On each internal (a_r, b_r), we choose a point c_r, $r = 1, 2, \dots, q$. Then we apply the construction in [10] on the interval (a_r, c_r) to obtain LC solutions u_{r1}, …, u_{rm_r}, and we apply the construction in [10] on the interval (c_r, b_r) to obtain LC solutions v_{r1}, …, v_{rn_r}. Using the LC solutions u_{r1}, …, u_{rm_r} and v_{r1}, …, v_{rn_r} for the left endpoint a_r and the right endpoint b_r of each of the q intervals $[a_r, b_r]$, $r = 1, 2, \dots, q$, we give the characterizations of all self-adjoint domains for singular symmetric operators with $q-1$ interior singular points or equivalently, all self-adjoint restrictions of the singular maximal operators in direct sum space in terms of the LC solutions of the $2r$ endpoints. These extensions yield "new" self-adjoint operators which are not merely direct sums of self-adjoint operators from the subintervals but involve interactions between the subintervals. These interactions are the interactions between singular endpoints. These will be illustrated in Section 4 with several examples.

2. Notations and preliminaries

Consider the even order symmetric differential expression

$$M = \sum_{j=0}^{n} p_j(x) D^j$$

over interval $I = (a, b)$, $-\infty < a < b < \infty$, where $p_j(x)$, $j = 0, 1, \dots, n$, are real-valued functions with some smooth and integrable conditions. We assume that there exists $q - 1 (1 \le q < +\infty)$ singular points of M in I.