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Existence of viscosity solution for a singular Hamilton–Jacobi equation

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Abstract. In this paper we study the existence of a singular Hamilton–Jacobi equation under the framework of viscosity solutions. The analysis is inspired by the arguments of [8] where a study of a model on dislocation dynamics was considered.

Keywords: Hamilton–Jacobi equations; Scalar conservation laws; Viscosity solutions; Entropy solutions; Dynamics of dislocation densities

AMS subject classifications: 70H20; 35L65; 49L25; 54C70; 74H20; 74H25

1. Introduction and statement of the results

In [8], the author analyses a one-dimensional system of partial differential equations modelling the dynamics of dislocation densities in crystals. Dislocations are topological defects within crystal structure that move under the submission of stress fields. Geometrically, each dislocation is characterised by a physical quantity called the Burgers vector, which is responsible for its orientation and magnitude. Dislocations are classified as being positive or negative due to the orientation of its Burgers vector, and they can move in certain crystallographic directions (see [7] for a physical study of dislocations).

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The system in [8] consists of a Hamilton–Jacobi (HJ) equation that is coupled to a diffusion equation, augmented with initial and Dirichlet boundary conditions. The main equation of concern is of the form

$$\kappa_t \kappa_x = \rho_t \rho_x, \tag{1.1}$$

where ρ satisfies (throughout all the papers)

$$\begin{cases} \rho_t = \rho_{xx} \text{ in } \quad \mathbb{R} \times (0, \infty), \\ \rho(\cdot, 0) = \rho^0 \in C_0^\infty(\mathbb{R}). \end{cases}$$

The above system is derived from the dynamics of dislocation densities $\theta^\pm$ that reads (see [6]):

$$\begin{cases} \theta_t^+ = \left[\left(\frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} \right) \theta^+ \right]_x \\ \theta_t^- = -\left[\left(\frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} \right) \theta^- \right]_x, \end{cases} \tag{1.2}$$

after taking an integrated form and making the following assumptions

$$\rho_x^\pm = \theta^\pm, \quad \rho = \rho^+ - \rho^- \quad \text{and} \quad \kappa = \rho^+ + \rho^-.$$

Various results regarding existence and uniqueness of viscosity solutions are established in [8]. Most of the results are obtained by bringing into service the connection between viscosity solutions and entropy solutions of conservation laws. To make this connection clear, we formally state the well-known result that if u is a viscosity solution of the Hamilton–Jacobi equation

$$u_t + H(x, t, u_x) = 0,$$

then the spatial derivative $v = u_x$ is an entropy solution of the scalar conservation laws

$$v_t + [H(x, t, v)]_x = 0.$$

The usual proof of this relation depends strongly on the known results about existence and uniqueness of the solutions of the two problems together with the convergence of the viscosity method (see [3,11,12]). Another proof of this relation could be found in [2] via the definition of viscosity/entropy inequalities, while a direct proof could also be found in [10] using the front tracking method. The case of dealing with a discontinuous Hamiltonian is treated in [13].

The relevant functional class for the Hamilton–Jacobi Eq. (1.1) is singled out to be the Lipschitz continuous functions possessing a suitable lower bound on the spatial gradient. The main complexity was in the singularity of the gradient of κ appearing in the Hamiltonian $H(x, t, \kappa_x) = \frac{f(x,t)}{\kappa_x}$ where $f(x, t) = \rho_t(x, t)\rho_x(x, t)$. Such singularity makes it difficult to define a solution of the equation in the case $\kappa_x = 0$. To overcome this difficulty, the author considered a suitable approximated problem:

$$\kappa_t^\epsilon \kappa_x^\epsilon = \rho_t \rho_x,$$

where it was shown that $\kappa_x^\epsilon > 0$ and therefore, by passing to the limit $\epsilon \to 0$, he obtained a solution of (1.1).