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On some numerical characteristics of operators

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Abstract. We investigate some numerical characteristics of Toeplitz operators including the numerical range, maximal numerical range and maximal Berezin set. Further, we establish an inequality for the Berezin number of an arbitrary operator on the Hardy–Hilbert space of the unit disc.

Keywords: Berezin symbol; Berezin number; Maximal numerical range; Maximal Berezin set; Toeplitz operator; Numerical range; Normal operator

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1. Introduction

In this article we investigate the so-called maximal numerical range in the sense of Stampfli [14] for some Toeplitz operators. We introduce the notion of maximal Berezin set for operators on a reproducing kernel Hilbert space (RKHS) and study some of its properties for the Toeplitz operators on the Hardy space $H^2(\mathbb{D})$. In particular, we focus on the model case of Toeplitz operators on the Hardy-Hilbert space on the unit disc. The Berezin number of an operator is also discussed.

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

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$$\|f\|_2^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{it})\right|^2 dt < +\infty.$$

The symbol $H^\infty = H^\infty(\mathbb{D})$ denotes the Banach algebra of functions bounded and analytic on the unit disc $\mathbb{D}$ equipped with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$. A function $\theta \in H^\infty$ for which $|\theta(\xi)| = 1$ almost everywhere in the unit circle $\mathbb{T}$ is called an inner function. It is convenient to establish a natural embedding of the space H^2 in the space $L^2 = L^2(\mathbb{T})$ by associating to each function $f \in H^2$ its radial boundary values $(bf)(\xi) := \lim_{r\to 1^-} f(r\xi)$, which exist for m-almost all $\xi \in \mathbb{T}$; where m is the normalized Lebesgue measure on $\mathbb{T}$. Then we have

$$H^2 = \left\{f \in L^2 : \hat{f}(n) = 0,\ n < 0\right\},$$

where $\hat{f}(n) := \int_{\mathbb{T}} \bar{\xi}^n f(\xi) dm(\xi)$ is the Fourier coefficient of the function f. For $\varphi \in L^\infty = L^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is the operator on H^2 defined by $T_\varphi f = P_+(\varphi f)$; here $P_+ : L^2(\mathbb{T}) \to H^2$ is the orthogonal projection (Riesz projector).

We shall use repeatedly the easy but useful fact that $T_\varphi^* \hat{k}_\lambda = \overline{\varphi(\lambda)} \hat{k}_\lambda$ for $\varphi \in H^\infty$; here $\hat{k}_\lambda$ is the normalized reproducing kernel for the Hardy space $H^2(\mathbb{D})$ (see Section 2).

2. On the maximal numerical range and maximal Berezin set

Recall that for the operator $T \in \mathcal{B}(H)$, (Banach algebra of all bounded linear operators on the Hilbert space H), Stampfli [14] defined the maximal numerical range as follows:

$$W_0(T) := \{\lambda \in \mathbb{C} : \langle Tx_n, x_n\rangle \to \lambda \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}.$$

When H is finite dimensional, it is easy to see that $W_0(T)$ corresponds to the numerical range produced by the maximal vectors (vectors x such that $\|x\| = 1$ and $\|Tx\| = \|T\|$). It is well known [14, Lemma 2] that the set $W_0(T)$ is nonempty, closed, convex, and contained in the closure of the usual numerical range

$$W(T) := \left\{\langle Tx, x\rangle : \|x\|_H = 1\right\}.$$

It is well known (see [7]) that $W(A)$ is a convex set whose closure contains the spectrum $\sigma(A)$ of A. If A is a normal operator, then the closure of $W(A)$ is the convex hull of $\sigma(A)$. Furthermore, it is also known that each extreme point of $W(A)$ is an eigenvalue of A.

Let $\mathcal{B}$ be a Banach algebra with the norm $\|.\|_{\mathcal{B}}$. A derivation on $\mathcal{B}$ is a linear map $\mathcal{D} : \mathcal{B} \to \mathcal{B}$ which satisfies

$$\mathcal{D}(ab) = a\mathcal{D}(b) + \mathcal{D}(a)b$$

for all $a, b \in \mathcal{B}$. If for a fixed a, $\mathcal{D}_a : b \to ab - ba$, then $\mathcal{D}_a$ is called an inner derivation. It is well known that every derivation on a von Neumann algebra or on a simple C^*-algebra is inner (see [8,12,13]). It is obvious that $\|\mathcal{D}_a\| \le 2\|a\|_{\mathcal{B}}$. Stampfli proved that (see [14, Theorem 4]) if $\mathcal{D}_T$ is a derivation on $\mathcal{B}(H)$, then $\|\mathcal{D}_T\| = 2dist(T, \mathbb{C}I)$, where $\mathbb{C}I$ denotes the set of all scalar operators λI $(\lambda \in \mathbb{C})$ on H. Stampfli also proved in terms of the maximal numerical range of T that $\|\mathcal{D}_T\| = 2\|T\|$ if and only if $0 \in W_0(T)$ (see [14, Theorem 4]). When $T = T_\varphi$, the Toeplitz operator defined on