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Applications of an identity of Andrews

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Abstract. In this paper, we give a bilateral form of an identity of Andrews, which is a generalization of the $_1\psi_1$ summation formula of Ramanujan. Using Andrews' identity, we deduce some new identities involving mock theta functions of second order and finally, we deduce some q-gamma, q-beta and eta function identities.

Keywords: q-Series; Mock theta functions; q-Gamma; q-Beta; Eta-functions

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1. Introduction and statement of results

In 1981, Andrews [2] has established the following identity

$$a^{-1}\sum_{n=0}^{\infty}\frac{(-q/a,AB/ab)_n}{(-B/a,-A/a)_{n+1}}(-b)^n-b^{-1}\sum_{n=0}^{\infty}\frac{(A,-aq/B)_n}{(-a,-A/b)_{n+1}}(-B/b)^n$$
$$=(a^{-1}-b^{-1})\frac{(A,B,bq/a,aq/b,q,AB/ab)_\infty}{(-b,-a,-A/b,-A/a,-B/b,-B/a)_\infty},\quad |b|,|B/b|<1, \tag{1.1}$$

where as usual

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$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \quad n \text{ is an integer},$$
$$(a_1,a_2,a_3,\ldots,a_m)_n = (a_1)_n(a_2)_n(a_3)_n\cdots(a_m)_n,$$
$$(a_1,a_2,a_3,\ldots,a_n)_\infty = (a_1)_\infty(a_2)_\infty(a_3)_\infty\cdots(a_n)_\infty.$$

This identity was proved using several summation and transformation formulae involving basic hypergeometric series. Putting $A = 0$, $a = -q/a$, $B = b/a$ and $b = -z$ in (1.1), we obtain the well-known ${}_1\psi_1$ summation formula of Ramanujan [9].

$$\sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}z^n = \frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty}{(z)_\infty(b)_\infty(b/az)_\infty(q/a)_\infty}, \tag{1.2}$$

As indicated by Andrews [2] in his paper, Agarwal [1] and Kang [6] have proved (1.1) using the three term transformation formula of ${}_3\phi_2$-series [3, Equation (III.33), p. 364]. Recently, Liu [7] obtained the following equivalent form of (1.1) using (1.2) along with Roger-Fine identity by employing q-exponential operators.

Theorem 1.3. *If* $|a|, |b| < 1$, *then*

$$a^{-1}\sum_{k=0}^{\infty}\frac{(-q/a,cd/ab)_k}{(-c/a,-d/a)_{k+1}}(-b)^k - b^{-1}\sum_{k=0}^{\infty}\frac{(-q/b,cd/ab)_k}{(-c/b,-d/b)_{k+1}}(-a)^k$$
$$= (a^{-1}-b^{-1})\frac{(q,aq/b,bq/a,c,d,cd/ab)_\infty}{(-a,-b,-c/a,-c/b,-d/a,-d/b)_\infty}. \tag{1.3}$$

One can recover (1.1) from (1.3) by using Sears transformation for ${}_3\phi_2$-series [3, Equation (III.9), p. 359].

The main objective of this paper is to give a bilateral form of (1.3). As applications of (1.3) we derive some new identities involving mock theta functions of second order and also some q-gamma, q-beta and eta-function identities.

The q-gamma function $\Gamma_q(x)$, was introduced by Thomae [11] and later by Jackson [5] as

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}, \quad 0<q<1. \tag{1.4}$$

q-Beta function is defined by

$$B_q(x,y) = (1-q)\sum_{n=0}^{\infty}\frac{(q^{n+1})_\infty}{(q^{n+y})_\infty}q^{nx}.$$

A relation between q-Beta function and q-gamma function is given by

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{1.5}$$

The Dedekind eta function is defined by