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The (exponential) multipartitional polynomials and polynomial sequences of multinomial type, Part I

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Abstract. We establish some formulas relating multipartitional polynomials to multinomial polynomials. They appear, respectively, as a natural extension of Bell polynomials and of polynomials of binomial type. Our results are illustrated by some comprehensive examples.

Keywords: Multipartitional polynomials; Polynomial sequences of multinomial type; Bell polynomials

1. Introduction

Recently, Mihoubi [4,5] studies the connection between Bell polynomials and binomial type sequences and deduces identities for complete and partial Bell polynomials.

As an extension of our previous results on bipartitional polynomials, see [1,6], we establish some connections between multipartitional polynomials and polynomials of multinomial type. They appear, respectively, as a natural extension of Bell polynomials and the polynomials of binomial type.

Let us introduce some definitions and notations.

We define the complete (exponential) multipartitional polynomial $A_{n_1,\ldots,n_r}$ in the variables $x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}$ as

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$$A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1},x_{0,\ldots,0,2},\ldots,x_{n_1,\ldots,n_r}) := \sum \frac{n_1!\cdots n_r!}{k_{0,\ldots,0,1}!k_{0,\ldots,0,2}!\cdots k_{n_1,\ldots,n_r}!} \times \left(\frac{x_{0,\ldots,0,1}}{0!\cdots 0!1!}\right)^{k_{0,\ldots,0,1}} \cdots \left(\frac{x_{n_1,\ldots,n_r}}{n_1!\cdots n_r!}\right)^{k_{n_1,\ldots,n_r}}, \quad (1)$$

where the summation is extended over all partitions of the multipartite number $(n_1,\ldots,n_r)$, that is, over all nonnegative integers $(k_{0,\ldots,0,1},\ldots,k_{n_1,\ldots,n_r})$ solution of the equations

$$\sum_{i_r=0}^{n_r}\cdots\sum_{i_1=0}^{n_1} i_j k_{i_1,\ldots,i_r} = n_j, \quad j=1,\ldots,r, \text{ with the convention } k_{0,\ldots,0}=0. \quad (2)$$

Also, for a given k integer, we define the partial (exponential) multipartitional polynomial of degree k : $B_{n_1,\ldots,n_r,k}$ in the variables $x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}$ as the sum

$$B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1},x_{0,\ldots,0,2},\ldots,x_{n_1,\ldots,n_r}) := \sum \frac{n_1!\cdots n_r!}{k_{0,\ldots,0,1}!k_{0,\ldots,0,2}!\ldots k_{n_1,\ldots,n_r}!} \times \left(\frac{x_{0,\ldots,0,1}}{0!\cdots 0!1!}\right)^{k_{0,\ldots,0,1}} \cdots \left(\frac{x_{n_1,\ldots,n_r}}{n_1!\cdots n_r!}\right)^{k_{n_1,\ldots,n_r}}, \quad (3)$$

where the summation is extended over all partitions of the multipartite number $(n_1,\ldots,n_r)$ into k parts, that is, over all nonnegative integers $(k_{0,\ldots,0,1},\ldots,k_{n_1,\ldots,n_r})$ solution of the equations

$$\begin{aligned} &\sum_{i_r=0}^{n_r}\cdots\sum_{i_1=0}^{n_1} i_j k_{i_1,\ldots,i_r} = n_j, \quad j=1,\ldots,r, \\ &\sum_{i_r=0}^{n_r}\cdots\sum_{i_1=0}^{n_1} k_{i_1,\ldots,i_r} = k. \end{aligned} \quad (4)$$

These polynomials generalize the partial and complete Bell polynomials, see [2,3,7,8], and for other recent results, see [4,5]. Some properties can be deduced from the above definitions, thus: for all real numbers α, β, γ we have

$$A_{n_1,\ldots,n_r}(\alpha_r x_{0,\ldots,0,1},\ldots,\alpha_1^{n_1}\cdots\alpha_r^{n_r} x_{n_1,\ldots,n_r}) = \alpha_1^{n_1}\cdots\alpha_r^{n_r} A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}), \quad (5)$$

$$B_{n_1,\ldots,n_r,k}(\beta\alpha_r x_{0,\ldots,0,1},\ldots,\beta\alpha_1^{n_1}\cdots\alpha_r^{n_r} x_{n_1,\ldots,n_r}) = \beta^k \alpha_1^{n_1}\cdots\alpha_r^{n_r} B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}). \quad (6)$$

The exponential generating functions for $A_{n_1,\ldots,n_r}$ and $B_{n_1,\ldots,n_r,k}$ are given by

$$\sum_{n_1,\ldots,n_r \geqslant 0} A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}) \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!} = \exp\left(\sum_{i_1+\cdots+i_r \geqslant 1} x_{i_1,\ldots,i_r} \frac{t_1^{i_1}}{i_1!}\cdots\frac{t_r^{i_r}}{i_r!}\right), \quad (7)$$

$$\sum_{n_1+\cdots+n_r \geqslant k} B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}) \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!} = \frac{1}{k!}\left(\sum_{i_1+\cdots+i_r \geqslant 1} x_{i_1,\ldots,i_r} \frac{t_1^{i_1}}{i_1!}\cdots\frac{t_r^{i_r}}{i_r!}\right)^k. \quad (8)$$

The polynomials of multinomial type $(f_{n_1,\ldots,n_r}(x))$ have the following property: $f_{0,\ldots,0}(x) := 1$ and