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# Multiples of repunits as sum of powers of ten

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**Abstract.** The sequence $\boldsymbol{P_{k,n}} = 1 + 10^{\boldsymbol{k}} + 10^{2\boldsymbol{k}} + \cdots + 10^{(\boldsymbol{n}-1)\boldsymbol{k}}$ can be used to generate infinitely many Smith numbers with the help of a set of suitable multipliers. We prove the existence of such a set, consisting of constant multiples of repunits, that generalizes to any value of $\boldsymbol{k} \geqslant 9$. This fact complements the earlier results which have been established for $\boldsymbol{k} \leqslant 9$.

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## 1. Introduction

A natural number $n$ is called a Smith number if $n$ is a composite for which the digital sum $S(n)$ equals the p-digit sum $S_p(n)$, where $S_p(n)$ is given by the digital sum of all the prime factors of $n$, counting multiplicity. For example, based on the factorization $636 = 2^2 \times 3 \times 53$, we have $S_p(636) = 2 + 2 + 3 + 5 + 3 = 15$. Since $S(636) = 6 + 3 + 6 = 15$, then $S(636) = S_p(636)$ and therefore, 636 is a Smith number.

Smith numbers were first introduced in 1982 by Wilansky [2]. We already know that Smith numbers are infinitely many—a fact first proved in 1987 by McDaniel [1]. In a quite recent publication [3], an alternate method for constructing Smith numbers was introduced, involving the sequence $P_{k,n}$ defined by

$$P_{k,n} = \sum_{i=0}^{n-1} 10^{ki}.$$

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The established fact [3, Theorem 9] can be restated as follows.

**Theorem 1.1.** *Let $k \geqslant 2$ be fixed, and let $M_k$ be a set of seven natural numbers with two conditions:*

1. *The set $\{S_p(t) \mid t \in M_k\}$ is a complete residue system modulo 7.*
2. *Every element $t \in M_k$ can be expressed as $t = \sum_{j=1}^{k} 10^{e_j}$, where the set $\{e_j \mid 1 \leqslant j \leqslant k\}$ is a complete residue system modulo $k$.*

*Then there exist infinitely many values of $n \geqslant 1$ for which the product*

$$9 \times P_{k,n} \times t_{k,n} \times 10^{f_{k,n}}$$

*is a Smith number for some element $t_{k,n} \in M_k$ and exponent $f_{k,n} \geqslant 0$.*

Following this result, the article continues with the construction of a set $M_k$ which satisfies the hypothesis of Theorem 1.1, for each $k = 2, 3, \ldots, 9$.

This paper is a response to the challenge to continue with the search for such $M_k$ for $k > 9$. Quite surprisingly we are able to give a relatively clean construction of $M_k$, which consists of seven constant multiples of the repunit $R_k = (10^k - 1)/9$, and which is valid for all $k \geqslant 9$ but not for lesser values of $k$.

## 2. Main results

**Theorem 2.1.** *Consider the repunit $R_k$ and let $m = 9 \cdot 10^a + 9 \cdot 10^b + 1$, where* $k > a > b > 0$. Then we can write $mR_k = \sum_{j=1}^{k} 10^{e_j}$ *such that the set $\{e_j \mid 1 \leqslant j \leqslant k\}$ serves as a complete residue system modulo $k$.*

**Proof.** Since we will be dealing with strings of repeated digits, let us agree on the following notation. By $(u_d)$, where $0 \leqslant u \leqslant 9$, we mean a string of $u$'s of length $d$ digits. In particular, when $d = 1$, we simply write $(u)$ instead of $(u_1)$. We also allow concatenation, e.g., the notation $(1_5,0,9_3,0_2,1)$ represents the number 111110999001.

Now let $A = 9 \cdot 10^a \cdot R_k$ and $B = 9 \cdot 10^b \cdot R_k$, hence $mR_k = A + B + R_k$. In order to help visualize how the addition $B + R_k$ is performed, we right-align the two strings and add columnwise, right to left, as follows.

$$\begin{aligned} R_k &= (1_k) = (1_{k-b}, 1_b), \\ B &= (9_k, 0_b) = (9_b, 9_{k-b}, 0_b), \\ B + R_k &= (1, 0_b, 1_{k-b-1}, 0, 1_b). \end{aligned}$$

Now with $A = (9_k, 0_a)$, we prepare the addition operation for $mR_k = A + (B + R_k)$ in a similar way:

$$\begin{aligned} B + R_k &= (1, 0_b, 1_{k-a}, 1_{a-b-1}, 0, 1_b), \\ A &= (9_{a-b-1}, 9, 9_b, 9_{k-a}, 0_{a-b-1}, 0, 0_b), \\ mR_k &= (1, 0_{a-b-1}, 1, 0_b, 1_{k-a-1}, 0, 1_{a-b-1}, 0, 1_b). \end{aligned}$$

(Note that in the case $a - b - 1 = 0$, each string of length $a - b - 1$ appearing above is simply nonexistent, and similarly for $k - a - 1$ if equals 0.)