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Double Hopf bifurcation in delay differential equations

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Abstract. The paper addresses the computation of elements of double Hopf bifurcation for retarded functional differential equations (FDEs) with parameters. We present an efficient method for computing, simultaneously, the coefficients of center manifolds and normal forms, in terms of the original FDEs, associated with the double Hopf singularity up to an arbitrary order. Finally, we apply our results to a nonlinear model with periodic delay. This shows the applicability of the methodology in the study of delay models arising in either natural or technological problems.

Keywords: Double Hopf; Delay; Bifurcation; Functional differential equation; Center manifold; Normal forms; Regenerative cutting tool

1. Introduction

The double Hopf bifurcation, which is readily located via linear stability analysis as the codimension-two point at which two pairs of complex conjugate eigenvalues have their real part simultaneously change sign, has associated very rich nonlinear dynamics in its neighborhood. Depending on the particulars of the system under consideration, there are around thirty different dynamical scenarios, divided into simple and difficult cases. Center manifold theory is of fundamental importance in the study of nonlinear dynamical systems when analyzing bifurcations of such a type. In fact, this theory allows us to reduce the study of a differential equation with delay near to a non-hyperbolic equilibrium point to that of an ordinary differential equation on a finite-dimensional invariant manifold. This

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approach is particularly interesting when the starting point is an infinite-dimensional problem, such as a functional, partial or an integro differential equation. In fact, the reduction forms also a qualitative simplification. Although center manifolds exist, they need not be unique, there are some points which must always be on any center manifold: all the center manifolds have the same Taylor expansion near the equilibrium up to their order of regularity; such an expansion may give an idea of the local dynamics near the steady state. Center manifolds and normal forms of such bifurcations have been discussed in [5,9] for ordinary differential equations. Center manifolds for Hopf and fold-Hopf bifurcations in FDEs with discrete delays have been studied in [6,7]. In [1,2], Faria and Magalhaès have considered the computation of coefficients of normal forms for functional differential equations (FDEs) of both Hopf and Bogdanov singularities. However, it is difficult to apply the method to compute the explicit expressions for these coefficients since it demands much more computation efforts for high-order normal forms.

The attention of this paper will be focused on the development of methodology and software for computing the center manifolds and normal forms of double Hopf bifurcation for general FDEs with parameters. According to the structure of linearized equation of a retarded system evaluated at an equilibrium, the case considered in this paper corresponds to two different pairs of purely imaginary eigenvalues.

The aim is twofold: the primary objective of this paper is to prove that the terms of a parameterized center manifold associated with double Hopf singularity up to the order of regularity for FDEs satisfy an efficient and explicit recursive formulas, and to give an efficient algorithm. The purpose of the second fold is to obtain, for the general situation of double Hopf bifurcation for FDEs, explicit formulas giving the coefficients of normal forms in terms of the coefficients of the original equation. This allows a simpler scheme that greatly saves computational time and computation memory. Furthermore, the obtained method combines center manifolds and normal form schemes into one step. To show the applicability and the efficiency of the method, we apply our algorithmic scheme to obtain normal forms of a nonlinear mechanical model with periodic delay. We achieve this by augmenting the explicit-time dependent delay terms as new state variables to the original equations of motions with appropriate initial conditions and applying the method of computing the center manifolds and normal forms obtained in the present paper. For the notation, background about the theory of FDEs and all needed results in the remainder of this paper, we follow [9]. However, we use $C_n = C([-r, 0], \mathbb{R}^n), r \geqslant 0$ since we need to work in realization spaces with different dimensions, depending on whether the parameters are incorporated or not incorporated in the realization space variables.

The paper is organized as follows: a computational scheme for a center manifold associated with double Hopf with parameters are given in Section 2. Normal forms for FDEs are derived, in terms of the original equation and the center manifold coefficients, in the same section. To illustrate our results, we study the bifurcation of a model arising from the mechanics in Section 3. Conclusions are drawn in Section 4.

2. Main results

In this section, we present our result concerning the computation of terms of center manifolds and normal forms for FDEs of the form