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A stochastic maximum principle in mean-field optimal control problems for jump diffusions

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Abstract. This paper is concerned with the study of a stochastic control problem, where the controlled system is described by a stochastic differential equation (SDE) driven by a Poisson random measure and an independent Brownian motion. The cost functional involves the mean of certain nonlinear functions of the state variable. The inclusion of this mean terms in the running and the final cost functions introduces a major difficulty when applying the dynamic programming principle. A key idea of solving the problem is to use the stochastic maximum principle method (SMP). In the first part of the paper, we focus on necessary optimality conditions while the control set is assumed to be convex. Then we prove that these conditions are in fact sufficient provided some convexity conditions are fulfilled. In the second part, the results are applied to solve the mean-variance portfolio selection problem in a jump setting.

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Keywords: Stochastic systems with jumps; Mean-field control problem; Stochastic maximum principle; Optimal control

1. Introduction

In this paper, we discuss stochastic control models which are driven by a stochastic differential equation with jumps, taking the following form

$$\begin{cases} dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t, u_t)dB_t + \int_Z \gamma(t, x_{t-}, u_t, z)\widetilde{N}(dt, dz), \\ x_0 = x. \end{cases} \tag{1.1}$$

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The objective is to minimize the following expected cost functional,

$$J(u) = \mathbb{E}\left[\int_0^T f(t, x_t, \mathbb{E}[\varphi(x_t)], u_t)dt + g(x_T, \mathbb{E}[\eta(x_T)])\right], \quad \text{for } u \in \mathcal{U}, \tag{1.2}$$

over the set of the admissible controls. The models where the coefficients f and g depend on the marginal probability law of the solution are not simple extensions from the mathematical point of view, but also provide interesting models in applications [1,16]. The typical example is the continuous-time Markowitz's mean-variance portfolio selection model, where one should minimize an objective function involving a quadratic function of the expectation, due to the variance term, see [3,15,2,22]. The main difficulty when facing a general mean-field controlled diffusion is that, the setting is non-Markovian, and hence, the dynamic programming and HJB techniques based on the law of iterated expectations on J do not hold in general. The stochastic maximum principle provides a powerful tool for handling this problem, see [2,17].

It is well known that the maximum principle for a stochastic optimal control problem provides necessary conditions of optimality obtained by duality theory, involving the so-called adjoint process, which solves a linear backward stochastic differential equation (BSDE in short). Some results that cover the controlled jump diffusion processes are discussed in [20,3,14,21,5,18]. Necessary and sufficient conditions of optimality for partial information control problems have been obtained in [3]. In [14] the sufficient maximum principle and the link with the dynamic programming principle are given. The second order stochastic maximum principle for optimal controls of nonlinear dynamics with jumps and convex state constraints was developed via spike variation method by Tang and Li [21]. These conditions are described in terms of two adjoint processes, which are linear backward SDEs [19]. Such equations have important applications in hedging problems, see [13]. Existence and uniqueness of solutions of BSDEs with jumps and nonlinear coefficients have been treated by Tang and Li [21], Barles et al. [4]. The link with integral-partial differential equations is studied in [4]. See Bouchard and Elie [7] for discrete time approximation of decoupled FBSDE with jumps. The notion of mean-field BSDE appears in [8,9]. Equations of this type are essentially generalizations of BSDEs, which allow the generator term to be an expectation of certain nonlinear function. In [8], a general notion of mean-field BSDE associated with a mean-field SDE is obtained in a natural way as a limit of some high dimensional system of FBSDEs governed by a d-dimensional Brownian motion, and influenced by positions of a large number of other particles. The study of mean-field BSDEs in a Markovian framework, associated with a mean-field SDE is given in [9]. By combining classical BSDE theory with specific arguments from mean-field BSDEs, it was shown that this mean-field BSDE describes the viscosity solution of a nonlocal PDE.

Control problems for jump diffusions have been treated in [10], to derive near-optimality conditions rather than exact optimality conditions. The control domain was supposed to be non convex, and the coefficients in the cost functional (1.2) do not depend explicitly on the mean-field terms. By using the spike variation method and Ekeland's variational principle, these conditions are obtained in terms of two adjoint processes.

Our main concern in the present paper is to derive the stochastic maximum principle for a stochastic control problem in which the mean-field terms appear in the cost