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Solving partial fractional differential equations using the $\mathcal{F}_A$-transform

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Abstract. In this article, we introduce the generalized Fourier transform ($\mathcal{F}_A$-transform) and derive an inversion formula and convolution product for this transform. Furthermore, the fundamental solutions of the single-order and distributed-order Cauchy type fractional diffusion equations are given by means of the appropriate $\mathcal{F}_A$-transform in terms of the Wright functions. Also, applicability of this transform for the explicit solution of the generalized Hilbert type singular integral equation is discussed.

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1. Introduction

We consider the Fourier-type integral transform called the $\mathcal{F}_A$-transform as follows

$$\mathcal{F}_A\{f(x);p\} = \int_{-\infty}^{\infty} A'(x)e^{-ipA(x)}f(x)dx, \quad p \in \mathbb{R}, \tag{1-1}$$

where $f(x)$ is piecewise continuous and absolutely integrable on $\mathbb{R}$ (i.e. $\int_{-\infty}^{\infty}|f(A^{-1}(x))|dx < \infty$) and the function $A(x)$ is strictly increasing function with asymptotic behaviors $\lim_{x\to\pm\infty}A(x) = \pm\infty$.

It is obvious that the $\mathcal{F}_A$-transform corresponds to the Fourier transform when it is $A(x) = x$.

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In the recent years, Mainardi et al. [15–19], Gorenflo et al. [11,12] and other researchers (e.g. [8,25]) have investigated on the diffusion- wave type equations and other equations of this type with constant coefficients containing fractional derivatives (in the Riemann–Liouville or Caputo senses of single and distributed orders) in time and/or in space to describe the models of anomalous transport in physics and engineering. They applied the joint transform method to boundary value problems to find the fundamental solutions of these equations in terms of higher transcendental functions such as Fox H-function, the Wright and the Mittag–Leffler functions. For the linear partial fractional differential equations (LPFDEs) with non-constant coefficients the existing integral transform methods (such as the joint Laplace-Fourier transform) are not applicable, therefore, importance of the $\mathcal{F}_A$-transform for solving some fractional-type equations with non-constant coefficients is emphasized. In this regard, we introduced a Laplace-type integral transform ($\mathcal{L}_A$-transform) for solving LPFDEs with non-constant coefficients, see [1–3].

In this work, we focus our attention on the LPFDEs with non-constant coefficients in terms of the ${}_A\delta_x$-derivatives which occur in physical phenomena such as *fractional diffusion with non-constant coefficients*, *distributed-order fractional diffusion* which can be easily solved by applying the $\mathcal{F}_A$-transform by choosing the appropriate $A(x)$. Furthermore, effectiveness of the $\mathcal{F}_A$-transform in solving the singular integral equations with convolution-type kernels is treated.

In Section 2, we derive a new inversion formula for the $\mathcal{F}_A$-transform in terms of the Fourier's integral. Two lemmas in the $\mathcal{F}_A$-transform of the ${}_A\delta_x$-derivatives and the convolution property are also established. These properties can be useful for obtaining the solutions of fractional diffusion with non-constant coefficients and distributed-order fractional diffusion.

In Section 3, we find the fundamental solution of the fractional diffusion equation on fractals introduced by Giona and Roman by applying the $\mathcal{F}_{\frac{x^{\beta+1}}{\beta+1}}$-transform ($\beta \geqslant 0$). These solutions can be expressed in terms of the higher transcendental functions of the Wright type.

In Section 4, Moshinskii's diffusion equation is generalized to a fractional diffusion equation of distributed order and by using the $\mathcal{F}_{\sinh^{-1}(x)}$- transform the fundamental solution of this equation is given as an integral representation in terms of the Laplace type integral.

In Section 5, applicability of the $\mathcal{F}_{x^{2n+1}}$-transform in solving the generalized Hilbert singular integral equation is discussed and finally the main conclusions are set in Section 6.

2. Elementary properties of the $\mathcal{F}_A$-transform

In this section, we establish some lemmas on the $\mathcal{F}_A$-transform which can be useful for solving LPFDEs. First, related to the classical Fourier transform and inverse Fourier transform

$$\mathcal{F}\{f(x);p\} = \int_{-\infty}^{\infty} e^{-ipx} f(x)dx, \tag{2-1}$$

$$\mathcal{F}^{-1}\{F(p);x\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx} F(p)dp, \tag{2-2}$$

we derive an inversion formula for the $\mathcal{F}_A$-transform.