Arab Journal of
Mathematical Sciences

Arab J Math Sci 19(1) (2013), 85–94

A remark on the existence of positive solutions for variable exponent elliptic systems

G.A. Afrouzi [a,*], S. Shakeri [a], N.T. Chung [b]

[a] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

[b] Department of Mathematics and Informatics, Quang Binh University, 312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Viet Nam

Received 12 March 2012; revised 21 July 2012; accepted 26 August 2012
Available online 5 September 2012

Abstract. In this article, we consider the system of differential equations

$$\begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)}[a(x)u^{\alpha(x)}v^{\gamma(x)} + h_1(x)] & \text{in } \Omega, \\ -\Delta_{q(x)} v = \lambda^{q(x)}[b(x)u^{\delta(x)}v^{\beta(x)} + h_2(x)] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary $\partial\Omega$, $1 < p(x), \ q(x) \in C^1(\overline{\Omega})$ are functions. The operator $-\Delta_{p(x)}\, u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian. When α, β, δ, γ satisfy some suitable conditions, we prove the existence of positive solution via sub-supersolution arguments without assuming sign conditions on the functions h_1 and h_2.

Mathematics subject classifications: 35J25; 35J60

Keywords: Positive solutions; Variable exponent elliptic systems; Sub-supersolutions

1. Introduction and preliminaries

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [1,2,14,20]). Many existence results have been obtained on this kind of problems, see for example [4,9,10,12,15–18]. In [6–8], Fan

* Corresponding author.
E-mail addresses: afrouzi@umz.ac.ir (G.A. Afrouzi), s.shakeri@umz.ac.ir (S. Shakeri), ntchung82@yahoo.com (N.T. Chung).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2012.08.002

et al. studied the regularity of solutions for differential equations with nonstandard $p(x)$-growth conditions.

In this paper, we mainly consider the existence of positive weak solutions for the system

$$\begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)}[a(x)u^{\alpha(x)}v^{\gamma(x)} + h_1(x)] & \text{in } \Omega, \\ -\Delta_{q(x)} v = \lambda^{q(x)}[b(x)u^{\delta(x)}v^{\beta(x)} + h_2(x)] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary $\partial\Omega, 1 < p(x), q(x) \in C^1(\overline{\Omega})$ are two functions. The operator $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian and the corresponding equation is called a variable exponent equation. Especially, if $p(x) \equiv p$ (a constant), (1.1) is the well-known (p,q)-Laplacian system and the corresponding equation is called a constant exponent equation. We have known that the existence of solutions for p-Laplacian elliptic systems has been intensively studied in the last decades, we refer to [3,11,13]. In [11], Hai et al. considered the existence of positive weak solutions for the p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda f(v) & \text{in } \Omega, \\ -\Delta_p v = \lambda g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

in which the first eigenfunction was used for constructing the subsolution of p-Laplacian problems. Under the condition

$$\lim_{u \to +\infty} \frac{f\left(M(g(u))^{1/(p-1)}\right)}{u^{p-1}} = 0 \text{ for all } M > 0,$$

the authors showed that the problem (1.2) has at least one positive solution provided that $\lambda > 0$ is large enough.

In [3], the author studied the existence and nonexistence of positive weak solution to the following quasilinear elliptic system

$$\begin{cases} -\Delta_p u = \lambda f(u,v) = \lambda u^{\alpha} v^{\gamma} & \text{in } \Omega, \\ -\Delta_q v = \lambda g(u,v) = \lambda u^{\delta} v^{\beta} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The first eigenfunction is used to construct the subsolution of problem (1.3), the main results are as follows:

(*i*) If $\alpha,\beta \geqslant 0, \gamma,\delta > 0$, $\theta = (p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$, then problem (1.3) has a positive weak solution for each $\lambda > 0$;
(*ii*) If $\theta = 0$ and $p\gamma = q(p - 1 - \alpha)$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, then problem (1.3) has no nontrivial nonnegative weak solution.