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The differential pencils with turning point on the half line

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Abstract. We investigate the inverse spectral problem of recovering pencils of second-order differential operators on the half-line with turning point. Using the asymptotic distribution of the Weyl function, we give a formulation of the inverse problem and prove the uniqueness theorem for the solution of the inverse problem.

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1. Introduction

We consider the differential equation

$$y''(x) + (\rho^2 R(x) + i\rho q_1(x) + q_0(x))y(x) = 0, \quad x \geqslant 0, \tag{1}$$

on the half-line with nonlinear dependence on the spectral parameter ρ. Let $a \geqslant 1$, and

$$R(x) = \begin{cases} -1, & 0 \leqslant x < a, \\ x - 1, & x \geqslant a, \end{cases} \tag{2}$$

i.e., the sign of the weight-function changes in an interior point $x = a$, which is called the turning point. The functions $q_j(x)$, $j = 0,1$, are complex-valued, $q_1(x)$ is absolutely continuous and $(1 + x)q_j^{(l)} \in L(0, \infty)$ for $0 \leqslant l \leqslant j \leqslant 1$.

Differential equations with spectral parameter and turning point arise in various problems of mathematics (see, for example, Tamarkin [7]). The classical Sturm–Liouville operators with turning points in the finite interval have been studied fairly completely in Freiling and Schneider [2]. Indefinite differential pencils produce

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significant qualitative modification in the investigation of the inverse problem. Some aspects of the inverse problem theory for differential pencils without turning points were studied in Khruslov and Shepelsky [5] and Yurko [8]. In Freiling and Yurko [3,4], the inverse problem was investigated for differential equations with m turning points. Also the inverse problem was investigated for differential pencils with turning point and nonlinear dependence on the spectral parameter in Yurko [11,12]. Here we investigate the uniqueness solution of the inverse problem with turning point when the weight-function changes in the linear form after turning point. As the main spectral characteristic for the boundary value problem, we introduce the so-called Weyl function.

In this paper, we will study the uniqueness theorem for Eq. (1) with spectral boundary condition. In Section 2, we determine the asymptotic forms of the solutions of Eq. (1) and derive characteristic function. In Section 3, we obtain the Weyl function and establish a formulation of the inverse problem. In Section 4, we prove the uniqueness theorem.

2. Primary results

We consider the boundary value problem L for Eq. (1) on the half-line with the boundary condition

$$U(y) := y'(0) + (\beta_1\rho + \beta_0)y(0) = 0, \tag{3}$$

where the coefficients β_1 and β_0 are complex numbers and $\beta_1 \neq \pm 1$. Denote $\Pi\pm := \{\rho: \pm Im\rho > 0\}$, $\Pi_0 := \{\rho: Im\rho = 0\}$. By the well-known method (see, Mennicken and Moller [6]; Tamarkin [7] and Freiling and Yurko [4]), we obtain a solution $e(x,\rho)$ of the Eq. (1) (which is called the Jost-type solution) with the following properties:

Theorem 2.1. *Eq. (1) has a unique solution $y = e(x,\rho)$, $\rho \in \Pi\pm$, $x \geqslant a$, with the following properties:*

1. *For each fixed $x \geqslant a$, the functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are holomorphic for $\rho \in \Pi_+$ and $\rho \in \Pi_-$ (i.e., they are piecewise holomorphic).*
2. *The functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are continuous for $x \geqslant a, \rho \in \overline{\Pi}_+$ and $\rho \in \overline{\Pi}_-$ (we differ the sides of the cut Π_0). In other words, for real ρ, there exist the finite limits*

$$e_{\pm}^{(\nu)}(x,\rho) = \lim_{z\to\rho,\ z\in\Pi_\pm} e^{(\nu)}(x,z).$$

Moreover, the functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are continuously differentiable with respect to $\rho \in \overline{\Pi}_+ \setminus \{0\}$ and $\rho \in \overline{\Pi}_- \setminus \{0\}$.

3. *For $x \to \infty, \rho \in \overline{\Pi}_\pm \setminus \{0\}$, $\nu = 0, 1$,*

$$e^{(\nu)}(x,\rho) = (\pm i\rho)^{\nu} R(x)^{\nu-\frac{1}{2}} exp(\pm(i\rho x - Q(x)))(1 + o(1)), \tag{4}$$

where $Q(x) = \frac{1}{2}\int_0^x q_1(t)dt$.

4. *For $|\rho| \to \infty, \rho \in \overline{\Pi}_\pm$, $\nu = 0, 1$, uniformly in $x \geqslant a$,*

$$e^{(\nu)}(x,\rho) = (\pm i\rho)^{\nu} R(x)^{\nu-\frac{1}{2}} exp(\pm(i\rho x - Q(x)))[1], \tag{5}$$