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ORIGINAL ARTICLE

Quantum states as realizations of groups ☆

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Abstract A quick review of some Lie algebras related to well-known groups is given. We start with the Heisenberg-Weyl algebra and after the definitions of the Fock states we give the definition of the coherent state of this group. This is followed by the exposition of the $SU(2)$ and $SU(1,1)$ algebras and their coherent states. From there we go on to describe the binomial states and their extensions as the finite dimensional pair coherent states and their nonlinear versions as realizations of the $SU(2)$ group. This is followed by considering the negative binomial states, the single mode and two-mode squeezed states and their variants as realizations of the $SU(1,1)$ group. Generation schemes based on physical systems are considered for some of these states.

☆ A tribute: This is a small tribute to the late Prof. Gamal M. Abd AlKader [1963–2009] one with whom I have a very fruitful and most interesting collaboration for almost two decades. His friendship and amicable personality, many of his colleagues and students as well as I really miss. May Allah accept him in His Mercy.

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1. Introduction

The use of the theory of groups in quantum mechanics started with the early days of that theory. The book titled The Theory of Groups and Quantum Mechanics of H. Weyl that was first published in German in 1928 (later its English translation appeared by Dover in 1950) (Weyl, 1950) is a standing witness to this. The Heisenberg-Weyl group was used at the very beginning for the study of some physical structures. Wider dimensions in various branches of physics such as high-energy physics, condensed matter, atomic and nuclear physics benefited greatly from the use of the group theory. With the advances in the field of quantum optics which began in the 60s, group theory started to infiltrate in this branch. Groups involving simple Lie algebras such as $SU(2)$ and $SU(1,1)$ and their simple generalization have been used to study different aspects in quantum optics.

In this article we review some states used in the field of quantum optics as realizations of the $SU(2)$ or $SU(1,1)$ groups. We start by some preliminaries about the annihilation and creation operators and the number operators which constitute the corner stones of the Heisenbeg-Weyl algebra, then their eigenstates are defined. The familiar algebras of the $SU(2)$ and $SU(1,1)$ are introduced. Then some of the quantum states which are realizations of the $SU(2)$ are reviewed in Section 3. Section 4 is devoted to states as realization of the $SU(1,1)$ group. Some comments are given about the generations of some of these states through physical processes.

2. Preliminaries

2.1. The harmonic oscillator

In the study of the harmonic oscillator, the following operators are introduced: the annihilation operator $\hat{a}$, the creation operator $\hat{a}^\dagger$, and the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. They satisfy the commutation relations

$$[a, a^\dagger] = I, \quad [n, a^\dagger] = a^\dagger, \quad [n, a] = -a. \tag{2.1}$$

The eigen-states $|n\rangle$ of the number operator $\hat{n}$ are called Fock states or number states. They satisfy

$$\hat{n}|n\rangle = n|n\rangle. \tag{2.2}$$

The non-negative integer n can be looked upon as the number of particles in the state. When $n = 0$ we call $|0\rangle$ the vacuum state with no particles present.

The operations of a and $a^\dagger$ on $|n\rangle$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2.3}$$

The states $\{|n\rangle\}$ form a complete set and resolve the unity

$$\sum_n |n\rangle\langle n| = I. \tag{2.4}$$