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Hyers–Ulam stability and discrete dichotomy for difference periodic systems

Constantin Buşe [a], Donal O'Regan [b], Olivia Saierli [c,*], Afshan Tabassum [d]

[a] *Politehnica University of Timişoara, Department of Mathematics, Timişoara, Romania*
[b] *National University of Ireland, School of Mathematics, Statistics and Applied Mathematics, Galway, Ireland*
[c] *Tibiscus University of Timişoara, Department of Computer Sciences, Timişoara, Romania*
[d] *Government College University, Abdus Salam School of Mathematical Sciences, (ASSMS), Lahore, Pakistan*

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ABSTRACT

Denote by $\mathbb{Z}_+$ the set of all nonnegative integer numbers. Let A_n be an $m \times m$ invertible q-periodic complex matrix, for all $n \in \mathbb{Z}_+$ and some positive integers m and q. First we prove that the discrete problem

$$x_{n+1} = A_n x_n, \quad x_n \in \mathbb{C}^m \tag{A_n}$$

is Hyers–Ulam stable if and only if the monodromy matrix T_q associated to the family $\mathcal{A} = \{A_n\}_{n\in\mathbb{Z}_+}$ possesses a discrete dichotomy.
Let (a_n), (b_n) be complex valued 2-periodic sequences. Consider the non-autonomous recurrence

$$z_{n+2} = a_n z_{n+1} + b_n z_n, \quad n \in \mathbb{Z}_+, z_n \in \mathbb{C} \tag{a_n, b_n}$$

and the matrix

$$A_n = \begin{pmatrix} 1 & 1 \\ a_n + b_n - 1 & a_n - 1 \end{pmatrix}, \quad n \in \mathbb{Z}_+.$$

* Corresponding author.
E-mail addresses: constantin.buse@upt.ro (C. Buşe), donal.oregan@nuigalway.ie (D. O'Regan), saierli_olivia@yahoo.com, osaierli@tibiscus.ro (O. Saierli), afshintabassum@gmail.com (A. Tabassum).

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We prove that the recurrence (a_n, b_n) is Hyers–Ulam stable if and only if the monodromy matrix $T_2 := A_1A_0$ has no eigenvalues on the unit circle.

The concept of exponential dichotomy for linear differential systems was introduced by O. Perron in 1930 [29]. Perron established the connection between the exponential dichotomy and the conditional stability of the system

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)) \tag{1}$$

in the context of finite dimensional spaces. Extensions of Perron's work to the general framework of infinite dimensional Banach spaces were obtained by M.G. Krein, R. Bellman, J.L. Massera and J.J. Schäffer in the period 1948–1966 (see [10,12,25]) and these authors studied the case when $A(t)$ are bounded linear operators. The case of unbounded coefficients was studied in [8,9,27,35] (see also the references therein).

The idea of passing from evolution equations to difference equations and vice versa has a long history that goes back to D. Henry [19].

The general framework of the stability problem for functional equations arose in 1940, due to a certain question posed by S.M. Ulam which was enunciated during a lecture which he delivered in front of the Club of Mathematics of the University of Wisconsin. In a particular case, this problem can be formulated as follows (see [36] for further details).

Given a metric group $(G, \cdot, d)$, a positive number ε, and a mapping $f : G \to G$ satisfying the inequality

$$d(f(x \cdot y), \quad f(x) \cdot f(y)) \leq \varepsilon \quad \textit{for all } x, y \in G, \tag{2}$$

does there exist a positive constant L and an automorphism g of G such that $d(f(x), g(x)) \leq L$ for all $x \in G$?

In 1941, D.H. Hyers [22] gave an affirmative answer to the Ulam Problem when G is the additive group of a real Banach space E. He showed that if $f : E \to E$ is a function verifying

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon, \quad \forall x, y \in E, \tag{3}$$

where ε is a given positive number, then the map

$$x \mapsto g(x) := \lim_{n\to\infty} 2^{-n} f(2^n x) : E \to E \tag{4}$$

is correctly defined and

$$\|g(x) - f(x)\| \leq \varepsilon \tag{5}$$