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On the rationality of Nagaraj–Seshadri moduli space

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ABSTRACT

We show that each of the irreducible components of moduli of rank 2 torsion-free sheaves with odd Euler characteristic over a reducible nodal curve is rational.

1. Introduction

Let X be a reducible nodal curve over an algebraically closed field k of characteristic 0 such that it is a union of two smooth irreducible components X_1 of genus $g_1 \geq 2$ and X_2 of genus $g_2 \geq 2$ meeting exactly at one node p. Let $\mathbf{a} = (a_1, a_2)$ be a tuple of positive rational numbers such that $a_1 + a_2 = 1$; we call this a polarisation on X. Let χ be an integer such that $a_1\chi$ is not an integer. In this setting it is a theorem of Nagaraj–Seshadri [6, Theorem 4.1] that the moduli space $M(2, \mathbf{a}, \chi)$ of semi-stable rank two torsion-free

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sheaves on X with Euler characteristic χ is a reduced, connected projective scheme with exactly two irreducible components, and when χ is odd, the moduli space is a union of two smooth varieties M_{12} and M_{21} intersecting transversally along a smooth divisor N.

Let $\xi = (L_1, L_2)$, where L_1 and L_2 are two invertible sheaves on X_1 and X_2 (of suitable degrees) respectively. Then in [6, Section 7] the analogue of a "fixed determinant moduli space" has been defined and we denote it by $M(2, \mathbf{a}, \chi, \xi)$. It is shown in ([17,1]) that when χ is odd and $a_1\chi$ is not an integer, $M(2, \mathbf{a}, \chi, \xi)$ is also a reduced, connected projective scheme with exactly two smooth components meeting transversally along a smooth divisor. The main result of this article is the following:

Theorem 1.1. *If gcd* $(\chi, 2) = 1$*, then both the irreducible components of* $M(2, \mathbf{a}, \chi, \xi)$ *are rational. In particular* $M(2, \mathbf{a}, \chi, \xi)$ *is rationally connected.*

Over a smooth projective curve of genus $g \geq 2$, the rationality of the moduli space was first proved by Tjurin [16, Theorem 14] in the rank 2 and odd degree case. When rank and degree are coprime this result was generalized by Newstead [9,10], King and Schofield [4] in higher order of generalities. It is still not known if the moduli space is rational or not in the non-coprime case, even for rank 2 and degree 0. In the non-smooth case, when the curve is irreducible and has any number of nodal singularities and genus ≥ 2, rationality in the coprime case was proved by Bhosle and Biswas [2, Theorem 3.7]. Over a reducible nodal curve X as described above it has been shown by Basu that each irreducible component of $M(2, \mathbf{a}, \chi, \xi)$ is unirational [1, Lemma 2.5]. Motivated by this result we go to the next step i.e. to prove rationality of each of these components. The proof of our result broadly follows the strategy of Newstead [9] but involves several technical difficulties.

It is well known that the moduli space of bundles over curves has a good specialization property, i.e. if a smooth projective curve Y specializes to a projective curve X with nodes as the only singularities, then the moduli space of vector bundles M_Y on Y specializes to the moduli space of torsion-free sheaves M_X on X [3,7,12,13]. It is known that rationality of projective varieties does not have a good specialization property, for example a family of cubic surface which is rational specializes to a non-rational surface which is birational to $E \times \mathbb{P}^1$ where E is a cubic curve. Our result shows that in the rank 2 and odd Euler characteristic case the moduli space of vector bundles gives an example of a family of rational varieties specializing to a rationally connected variety with two irreducible rational components. We hope that Theorem 1.1 will be useful in the study of degeneration of higher dimensional smooth projective algebraic varieties.

Further it will be interesting to see if Theorem 1.1 can be generalized to a more general situation i.e. if the underlying curve C has more than 2 components together with more than one node. In such a general situation the moduli space of semistable torsion-free sheaves (arbitrary rank) has been constructed by Seshadri (see [13, Chapter VII]). In particular when C is a tree like curve without any rational components, then the number of components of the moduli space and inequalities involving Euler characteristics