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Equivariant Abelian principal bundles on nonsingular toric varieties

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ABSTRACT

We give a classification of the holomorphic (resp. algebraic) torus equivariant principal G-bundles on a nonsingular toric variety X when G is an Abelian, closed, holomorphic (resp. algebraic) subgroup of the complex general linear group. We prove that any such bundle splits, that is, admits a reduction of structure group to a torus. We give an explicit parametrization of the isomorphism classes of such bundles for a large family of G when X is complete.

1. Introduction

Denote the algebraic torus $(\mathbb{C}^*)^n$ by T. A T-equivariant principal G-bundle on a complex manifold X is a locally trivial, principal G-bundle $\pi : \mathcal{E} \to X$ such that $\mathcal{E}$ and X are left T-spaces, the map π is T-equivariant and the actions of T and G commute:

$$t(e \cdot g) = (te) \cdot g \text{ for all } t \in T,\ g \in G \text{ and } e \in \mathcal{E}.$$

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If, in addition, the bundle $\pi : \mathcal{E} \to X$ and the actions of T and G are holomorphic, we say that $\mathcal{E}$ is a holomorphic T-equivariant principal G-bundle.

Let X_Ξ be a nonsingular toric variety of dimension n corresponding to a fan Ξ. Denote the set of d-dimensional cones in Ξ by $\Xi(d)$. For a cone σ in Ξ, denote the corresponding affine variety by X_σ and the corresponding T-orbit by O_σ. Note that each orbit O_σ has a natural group structure and the principal orbit O is identified with T (see [7, Proposition 1.6]). Let T_σ denote the stabilizer of any point in O_σ. Then T admits a decomposition, $T \cong T_\sigma \times O_\sigma$. Let $\pi_\sigma : T \to T_\sigma$ be the associated projection.

Let G be an Abelian, closed, holomorphic subgroup of $GL_k(\mathbb{C})$ for some positive integer k. (Note that an Abelian linear algebraic group satisfies this hypothesis, but not every Abelian, closed, holomorphic subgroup of $GL_k(\mathbb{C})$ is algebraic.) Then our main theorem is the following (same as Theorem 4.1).

Theorem 1.1. *The isomorphism classes of holomorphic T-equivariant principal G-bundles on X_Ξ are in one-to-one correspondence with collections of holomorphic group homomorphisms $\{\rho_\sigma : T_\sigma \to G \mid \sigma$ is a maximal cone in $\Xi\}$ which satisfy the extension condition: Each $(\rho_\tau \circ \pi_\tau)(\rho_\sigma \circ \pi_\sigma)^{-1}$ extends to a G-valued holomorphic function over $X_\sigma \cap X_\tau$.*

A similar classification for algebraic T-equivariant principal G-bundles over X_Ξ is given in Theorem 4.3. In this case G is assumed to be an Abelian linear algebraic group. However, using the classification of Abelian subgroups of the general linear group in [1], one can show that any holomorphic homomorphism $\rho : T \to G$ is algebraic. This follows from Lemma 5.2. Therefore, the isomorphism classes of algebraic and holomorphic T-equivariant principal G-bundles coincide when G is an Abelian linear algebraic group.

These theorems provide a partial analogue to Klyachko's classification of vector bundles on toric varieties in [6]. In particular, we show that any T-equivariant principal G-bundle over a nonsingular affine toric variety is trivializable (see Lemma 3.4). The main tool used by Klyachko for the local classification (i.e. classification on affine toric variety) is representation theory. In analogous situation, our main tool is complex analysis. However, our approach does not work when G is not Abelian since the proof of Theorem 3.2 relies heavily on this assumption. In this article, we are restricted to the case of nonsingular toric variety because we use Oka–Grauert theory [4].

At this time we should point out the work of Heinzner and Kutzschebauch [5]. Their work is in much more general setting where T is any complex reductive group and the structure group G is a complex Lie group. They prove the T-equivariant version of Grauert's Oka principle. In particular they show that any T-equivariant principal G-bundle over $\mathbb{C}^n$ is equivariantly isomorphic to $\mathbb{C}^n \times G$ where the latter has diagonal T-action corresponding to some homomorphism $T \to G$. Hence as a special case we get our Theorem 3.2. In other words, we give an alternative simple proof of their theorem in a very special case.

As a corollary of the main theorem(s), we prove that any such T-equivariant principal G-bundle splits, that is, admits a reduction of structure group to the intersection of G