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Degrees of maps between locally symmetric spaces

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ABSTRACT

Let X be a locally symmetric space $\Gamma\backslash G/K$ where G is a connected non-compact semisimple real Lie group with trivial centre, K is a maximal compact subgroup of G, and $\Gamma \subset G$ is a torsion-free irreducible lattice in G. Let $Y = \Lambda\backslash H/L$ be another such space having the same dimension as X. Suppose that real rank of G is at least 2. We show that any $f : X \to Y$ is either null-homotopic or it is homotopic to a covering projection of degree an integer that depends only on Γ and Λ. As a corollary we obtain that the set $[X, Y]$ of homotopy classes of maps from X to Y is finite.
We obtain results on the (non-)existence of orientation reversing diffeomorphisms on X as well as the fixed point property for X.

1. Introduction

Let G be a connected non-compact semisimple real Lie group with trivial centre and without compact factors and let $K \subset G$ be a maximal compact subgroup. Then G/K is a Riemannian globally symmetric space which is contractible. If Γ is a torsion-free lattice in G, then the locally symmetric space $X = \Gamma\backslash G/K$ (of non-compact type) is a manifold which is an Eilenberg–MacLane space $K(\Gamma, 1)$. Locally symmetric spaces are

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fundamental objects which arise in different areas of mathematics such as geometry, Lie groups, representation theory, number theory as well as topology. In this article we study the problem of homotopy classification of maps between two such spaces.

Our main result is the following. Recall that a lattice $\Gamma \subset G$ in a connected semisimple real Lie group is called *irreducible* if its image in G/N under the projection $G \to G/N$ is dense for any non-compact normal subgroup N of G. (This definition is weaker than the one given in [37]; the two definitions agree when G has no non-trivial compact factors.)

Let $f : M \to N$ be any continuous map between two oriented connected manifolds of the same dimension n. Recall that if M and N are compact, then $\deg(f) \in \mathbb{Z}$ is defined by $f_*(\mu_M) = \deg(f).\mu_N$ where $f_* : H_n(M;\mathbb{Z}) = \mathbb{Z}\mu_M \to \mathbb{Z}\mu_N = H_n(N;\mathbb{Z})$ and μ_M, μ_N are the fundamental classes of M and N respectively. If M and N are non-compact and if f is *proper*, then $\deg(f) \in \mathbb{Z}$ is defined in an analogous manner using compactly supported cohomology.

Theorem 1.1. *Let G, H be connected semisimple Lie groups with trivial centre and without compact factors and let K, L be maximal compact subgroups of G and H respectively. Suppose that the (real) rank of G is at least 2 and that $\dim G/K \geq \dim H/L$. Let Γ be an irreducible torsionless lattice in G and let Λ be any torsionless lattice in H. There exists a non-negative integer $\delta = \delta(\Gamma, \Lambda)$ such that the following holds: Any continuous map $f : \Gamma\backslash G/K \to \Lambda\backslash H/L$ is either null-homotopic or it is homotopic to a proper map g such that $\deg(g) = \pm\delta$.*

It will be shown that $\delta(\Gamma, \Lambda)$ equals $[\Lambda : \Lambda']$, the index of $\Lambda' \subset \Lambda$, where Λ' is *any* finite index subgroup of Λ isomorphic to Γ. If there is no such subgroup, then $\delta(\Gamma, \Lambda) = 0$. The assertion in the above theorem that f is homotopic to a covering projection if it is not null-homotopic follows by a straightforward argument using the rigidity theorem and the Margulis' normal subgroup theorem. By comparing the volumes of the locally symmetric spaces it can be seen that $\delta(\Gamma, \Lambda)$ is *independent* of the continuous map f. When $\delta > 0$, both δ and $-\delta$ occur as degrees of maps as in the theorem only if G/K admits an orientation reversing isometry. See Remark 3.2. We classify, almost completely, all globally symmetric spaces G/K with G simple, such that for *any* lattice $\Gamma \subset G$, the locally symmetric space $X = \Gamma\backslash G/K$ does not admit an orientation reversing isometry. The only cases which remain unsettled are certain G/K where G is exceptional.

We obtain the following result as a corollary.

Theorem 1.2. *Let $X = \Gamma\backslash G/K, Y = \Lambda\backslash H/L$ where G, H, Γ, Λ satisfy the hypotheses Theorem 1.1. Then the set $[X, Y]$ of all (free) homotopy classes of maps from X to Y is finite.*

Theorem 1.1 is not valid when G has rank 1. If Γ is such that its abelianization is infinite, then there exist infinitely many non-trivial homomorphisms $\Gamma \to \Lambda$ with image an infinite cyclic group. Using this, it is easily seen that there are infinitely many pairwise