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Transportation-cost inequalities on path spaces over manifolds carrying geometric flows

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ABSTRACT

Let $L_t := \Delta_t + Z_t$ for a $C^{1,1}$-vector field Z on a differential manifold M possibly with a boundary ∂M, where Δ_t is the Laplacian operator induced by a time dependent metric g_t differentiable in $t \in [0, T_c)$. In this article, by constructing suitable coupling, transportation-cost inequalities on the path space of the (reflecting if $\partial M \neq \varnothing$) diffusion process generated by L_t are proved to be equivalent to a new curvature lower bound condition and the convexity of the geometric flow (i.e., the boundary keeps convex). Some of them are further extended to non-convex flows by using conformal changes of the flows. As an application, these results are applied to the Ricci flow with the umbilic boundary.

1. Introduction

In this study, we aim to establish transportation-cost inequalities associated with the uniform distance, which is on the path space of (reflecting) diffusion processes over manifolds carrying a complete geometric flow. More precisely, our base manifold is a d-dimensional differential manifold M possibly with boundary ∂M equipped with

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a family of complete Riemannian metrics $(g_t)_{t\in[0,T_c)}$ for some $T_c \in (0,\infty]$, which is C^1 in t. Let ∇^t and Ric_t be, respectively, the Levi-Civita connection and the Ricci curvature tensor associated with the metric g_t. For simplicity, we take the notation: for $X, Y \in TM$,

$$\mathcal{R}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y\right\rangle_t - \frac{1}{2}\partial_t g_t(X,Y),$$

where Z_t is a $C^{1,1}$-vector field and $\langle\cdot,\cdot\rangle_t := g_t(\cdot,\cdot)$. Define the second fundamental form of the boundary with respect to g_t by

$$\mathrm{II}_t(X,Y) = -\left\langle \nabla_X^t N_t, Y\right\rangle_t, \quad \text{for all} \;\; X,Y \in T\partial M,$$

where N_t is the inward unit normal vector field of the boundary associated with the metric g_t and $T\partial M$ is the tangent space of ∂M. If $\mathrm{II}_t \geq 0$ for all $t \in [0,T_c)$, i.e., ∂M keeps convex for all $t \in [0,T_c)$, then we call $\{g_t\}$ a convex flow.

Consider the elliptic operator $L_t := \Delta_t + Z_t$ on $[0,T_c)\times M$, where Δ_t is the Laplacian operator with respect to the metric g_t and Z is a $C^{1,1}$-vector field. Let $\mu \in \mathscr{P}(M)$, where $\mathscr{P}(M)$ is the set including all probability measures on M. A (reflecting) diffusion process X^μ, generated by L_t with initial distribution μ, can be constructed as in [4]. Assume that X_t^μ is non-explosive before time T_c, by a similar discussion as in [5, Corollary 2.2], which is the case if

$$\mathcal{R}_t^Z \geq K(t), \text{ for some } K \in C([0,T_c)) \text{ and } \; \mathrm{II}_t \geq 0 \,(\text{if } \partial M \neq \varnothing), \; t \in [0,T_c). \tag{1.1}$$

When $\mu = \delta_x$, we simply denote $X_t^{\delta_x} = X_t^x$. Moreover, by [4], we know that X_t^x solves the following equation

$$\mathrm{d}X_t = \sqrt{2}u_t \circ \mathrm{d}B_t + Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, \;\; X_0 = x, \tag{1.2}$$

where $B_t := (B_t^1, B_t^2, \cdots, B_t^d)$ is a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$, u_t is the horizontal lift process of X_t and l_t is an increasing process supported on $\{t \geq 0 : X_t \in \partial M\}$. Note that if $\partial M = \varnothing$, then $l_t = 0$.

Given $\mu \in \mathscr{P}(M)$ and $0 \leq S < T < T_c$, let $\Pi_\mu^{[S,T]}$ be the distribution of $X_{[S,T]} := \{X_t : t \in [S,T]\}$ with initial law μ at time S. Then $\Pi_\mu^{[S,T]}$ is a probability measure on $W^{[S,T]} := C([S,T];M)$ with σ-field $\mathcal{F}^{[S,T]}$ induced by cylindrically measurable functions. When $S = 0$, we simply denote $\Pi_\mu^T := \Pi_\mu^{[0,T]}$ and $W^T := W^{[0,T]}$. Our aim is to establish transportation-cost inequalities for $\Pi_\mu^{[S,T]}$ under some new curvature conditions, which may include the influence from the time changing of the metric.

Transportation-cost inequality was first introduced by Talagrand [15] in 1996 to bound from above the L^2-Wasserstein distance to the standard Gaussian measure on $\mathbb{R}^d$ by the relative entropy. This inequality has been extended to distributions on finite- and infinite-dimensional spaces. In particular, this inequality was established on the path space of