

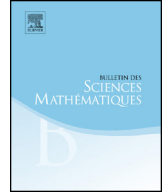


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# Transportation-cost inequalities on path spaces over manifolds carrying geometric flows



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## ABSTRACT

Let  $L_t := \Delta_t + Z_t$  for a  $C^{1,1}$ -vector field  $Z$  on a differential manifold  $M$  possibly with a boundary  $\partial M$ , where  $\Delta_t$  is the Laplacian operator induced by a time dependent metric  $g_t$  differentiable in  $t \in [0, T_c)$ . In this article, by constructing suitable coupling, transportation-cost inequalities on the path space of the (reflecting if  $\partial M \neq \emptyset$ ) diffusion process generated by  $L_t$  are proved to be equivalent to a new curvature lower bound condition and the convexity of the geometric flow (i.e., the boundary keeps convex). Some of them are further extended to non-convex flows by using conformal changes of the flows. As an application, these results are applied to the Ricci flow with the umbilic boundary.

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## 1. Introduction

In this study, we aim to establish transportation-cost inequalities associated with the uniform distance, which is on the path space of (reflecting) diffusion processes over manifolds carrying a complete geometric flow. More precisely, our base manifold is a  $d$ -dimensional differential manifold  $M$  possibly with boundary  $\partial M$  equipped with

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a family of complete Riemannian metrics  $(g_t)_{t \in [0, T_c]}$  for some  $T_c \in (0, \infty]$ , which is  $C^1$  in  $t$ . Let  $\nabla^t$  and  $\text{Ric}_t$  be, respectively, the Levi-Civita connection and the Ricci curvature tensor associated with the metric  $g_t$ . For simplicity, we take the notation: for  $X, Y \in TM$ ,

$$\mathcal{R}_t^Z(X, Y) := \text{Ric}_t(X, Y) - \langle \nabla_X^t Z_t, Y \rangle_t - \frac{1}{2} \partial_t g_t(X, Y),$$

where  $Z_t$  is a  $C^{1,1}$ -vector field and  $\langle \cdot, \cdot \rangle_t := g_t(\cdot, \cdot)$ . Define the second fundamental form of the boundary with respect to  $g_t$  by

$$\text{II}_t(X, Y) = - \langle \nabla_X^t N_t, Y \rangle_t, \quad \text{for all } X, Y \in T\partial M,$$

where  $N_t$  is the inward unit normal vector field of the boundary associated with the metric  $g_t$  and  $T\partial M$  is the tangent space of  $\partial M$ . If  $\text{II}_t \geq 0$  for all  $t \in [0, T_c)$ , i.e.,  $\partial M$  keeps convex for all  $t \in [0, T_c)$ , then we call  $\{g_t\}$  a convex flow.

Consider the elliptic operator  $L_t := \Delta_t + Z_t$  on  $[0, T_c) \times M$ , where  $\Delta_t$  is the Laplacian operator with respect to the metric  $g_t$  and  $Z$  is a  $C^{1,1}$ -vector field. Let  $\mu \in \mathcal{P}(M)$ , where  $\mathcal{P}(M)$  is the set including all probability measures on  $M$ . A (reflecting) diffusion process  $X^\mu$ , generated by  $L_t$  with initial distribution  $\mu$ , can be constructed as in [4]. Assume that  $X_t^\mu$  is non-explosive before time  $T_c$ , by a similar discussion as in [5, Corollary 2.2], which is the case if

$$\mathcal{R}_t^Z \geq K(t), \text{ for some } K \in C([0, T_c)) \text{ and } \text{II}_t \geq 0 \text{ (if } \partial M \neq \emptyset \text{), } t \in [0, T_c). \tag{1.1}$$

When  $\mu = \delta_x$ , we simply denote  $X_t^{\delta_x} = X_t^x$ . Moreover, by [4], we know that  $X_t^x$  solves the following equation

$$dX_t = \sqrt{2}u_t \circ dB_t + Z_t(X_t)dt + N_t(X_t)dl_t, \quad X_0 = x, \tag{1.2}$$

where  $B_t := (B_t^1, B_t^2, \dots, B_t^d)$  is a  $\mathbb{R}^d$ -valued Brownian motion on a complete filtered probability space  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  with the natural filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ ,  $u_t$  is the horizontal lift process of  $X_t$  and  $l_t$  is an increasing process supported on  $\{t \geq 0 : X_t \in \partial M\}$ . Note that if  $\partial M = \emptyset$ , then  $l_t = 0$ .

Given  $\mu \in \mathcal{P}(M)$  and  $0 \leq S < T < T_c$ , let  $\Pi_\mu^{[S, T]}$  be the distribution of  $X_{[S, T]} := \{X_t : t \in [S, T]\}$  with initial law  $\mu$  at time  $S$ . Then  $\Pi_\mu^{[S, T]}$  is a probability measure on  $W^{[S, T]} := C([S, T]; M)$  with  $\sigma$ -field  $\mathcal{F}^{[S, T]}$  induced by cylindrically measurable functions. When  $S = 0$ , we simply denote  $\Pi_\mu^T := \Pi_\mu^{[0, T]}$  and  $W^T := W^{[0, T]}$ . Our aim is to establish transportation-cost inequalities for  $\Pi_\mu^{[S, T]}$  under some new curvature conditions, which may include the influence from the time changing of the metric.

Transportation-cost inequality was first introduced by Talagrand [15] in 1996 to bound from above the  $L^2$ -Wasserstein distance to the standard Gaussian measure on  $\mathbb{R}^d$  by the relative entropy. This inequality has been extended to distributions on finite- and infinite-dimensional spaces. In particular, this inequality was established on the path space of

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