Bull. Sci. math. 140 (2016) 575–614

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

On the three-dimensional magnetohydrodynamics system in scaling-invariant spaces

Kazuo Yamazaki

Department of Mathematics, Washington State University, Pullman, WA 99164-3113, USA

A R T I C L E   I N F O

Article history:
Received 20 July 2015
Available online 3 September 2015

MSC:
35B65
35Q35
35Q86

Keywords:
Navier–Stokes equations
Magnetohydrodynamics system
Global regularity
Anisotropic Littlewood–Paley theory

A B S T R A C T

We study the criterion for the velocity and magnetic vector fields that solve the three-dimensional magnetohydrodynamics system, given any initial data sufficiently smooth, to experience a finite-time blowup. Following the work of Chemin and Zhang [12] and making use of the structure of the system, we obtain a criterion that is imposed on the magnetic vector field and only one of the three components of the velocity vector field, both in scaling-invariant spaces.

1. Introduction and statement of results

We study the following magnetohydrodynamics system in $\mathbb{R}^3$:

$$\frac{du}{dt} + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla\pi = \nu\Delta u, \tag{1a}$$

$$\frac{db}{dt} + (u \cdot \nabla)b - (b \cdot \nabla)u = \eta\Delta b, \tag{1b}$$

$$\nabla \cdot u = \nabla \cdot b = 0, \qquad (u, b)(x, 0) = (u_0, b_0)(x), \tag{1c}$$

E-mail address: kyamazaki@math.wsu.edu.

http://dx.doi.org/10.1016/j.bulsci.2015.08.003

where $u, b : \mathbb{R}^3 \times \mathbb{R}^+ \mapsto \mathbb{R}^3$ represent the velocity and magnetic vector fields respectively while $\pi : \mathbb{R}^3 \times \mathbb{R}^+ \mapsto \mathbb{R}$ is the scalar pressure field, $\nu \geq 0$ is the viscosity and $\eta \geq 0$ is the magnetic diffusivity. Without loss of generality, hereafter we assume $\nu = \eta = 1$ and also write ∂_t for $\frac{d}{dt}$ and ∂_i for $\frac{d}{dx_i}$, $i = 1, 2, 3$, $x = (x_1, x_2, x_3)$. Let us also set for any three-dimensional vector field $f = (f^1, f^2, f^3)$, $f^h \triangleq (f^1, f^2, 0)$,

$$\Omega \triangleq \nabla \times u, \qquad j \triangleq \nabla \times b, \qquad \omega \triangleq \Omega \cdot e^3, \qquad d \triangleq j \cdot e^3 \quad \text{where } e^3 \triangleq (0, 0, 1).$$

When $b \equiv 0$, (1a)–(1c) recovers the Navier–Stokes equations (NSE), for which the question of whether a smooth local solution can preserve its regularity for all time remains unknown. The analogous problem for the MHD system (1a)–(1c) remains just as difficult, if not more. One of the sources of the difficulty of the global regularity issue of the MHD system (1a)–(1c) may be traced back to the rescaling and its known bounded quantities. It can be shown that if $(u, b)(x, t)$ solves the system (1a)–(1c), then so does $(u_\lambda, b_\lambda)(x, t) \triangleq \lambda(u, b)(\lambda x, \lambda^2 t)$ while $\|u_\lambda(x, t)\|_{L^2}^2 + \|b_\lambda(x, t)\|_{L^2}^2 = \lambda^{-1}(\|u(x, \lambda^2 t)\|_{L^2}^2 + \|b(x, \lambda^2 t)\|_{L^2}^2)$.

In two-dimensional case, both the NSE and the MHD system, if $\nu, \eta > 0$, admit a unique global smooth solution starting from any data sufficiently smooth (cf. [22,24]). Due to the difficulty in the three-dimensional case, much effort has been devoted to provide regularity and blow-up criterion some of which we review now.

In [25], the author initiated important research direction of regularity criterion which led to, along with others such as [13], that if a weak solution u of the three-dimensional NSE with $\nu > 0$ satisfies

$$u \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{r} \leq 1, \ p \in [3, \infty],$$

then u is smooth. Among many other results, in [3] it was shown that if u solves the NSE with $\nu > 0$ and

$$\nabla u \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{r} \leq 2, \quad 1 < r \leq 3,$$

then u is a regular solution (cf. also [2,15]). We emphasize that the norm $\|\cdot\|_{L_T^r L_x^p}$ and $\|\cdot\|_{L_T^r \dot{W}_x^{1,p}}$ are both invariant under the scalings of the solutions to the NSE and the MHD system precisely when $\frac{3}{p} + \frac{2}{r} = 1$, $\frac{3}{p} + \frac{2}{r} = 2$ respectively. For the MHD system, e.g. the author in [26] showed that if $\nabla u, \nabla b \in L^4(0, T; L^2(\mathbb{R}^3))$, then no singularity occurs in $[0, T]$. Moreover, the work in [6] in particular showed that if $[0, T^*)$ is the maximal interval of existence of smooth solution and $T^* < \infty$, then

$$\int_0^{T^*} \|\Omega\|_{L^\infty} + \|j\|_{L^\infty} d\tau = \infty.$$

We note that the authors in [16,36] independently realized that in particular the criterion for the solution to the MHD system may be reduced to just u, dropping condition on