Bull. Sci. math. 140 (2016) 629–637

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

On the fixed points set of differential systems reversibilities

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ARTICLE INFO

Article history:
Received 6 October 2015
Available online 22 March 2016

Keywords:
ODE
Center
Reversibility
Divergence

ABSTRACT

We extend a result proved in [7] for mirror symmetries of planar systems to measure-preserving non-linear reversibilities of n-dimensional systems, dropping the analyticity and nondegeneracy conditions.

1. Introduction

Let us consider a planar differential system

$$\dot{z} = F(z), \tag{1}$$

where $F(z) = (F_1(z), F_2(z)) \in C^1(\Omega, \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$ open and connected, $z \in \Omega$. We denote by $\phi(t, z)$ the local flow defined by (1) in Ω. We assume the origin O to be a critical point of (1). One of the classical problems considered in the study of such systems is the so-called *center-focus* problem for a rotation point. It consists in distinguishing among the two following possibilities,

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http://dx.doi.org/10.1016/j.bulsci.2016.03.007
0007-4497/

- O is an attractor;
- O is an accumulation point of cycles, in particular a center.

Poincaré and Lyapunov developed a procedure that applies to analytic systems with a non-degenerate critical point O, allowing to discern among the only possible cases for analytic systems, i.e. a center or a focus.

If the system is non-analytic or the critical point is degenerate, in order to prove that a system has a center at O one can either look for a first integral with an isolated extremum at O, or apply a symmetry argument first used by Poincaré, looking for a mirror symmetry of the solution set. The symmetry method is easy to apply if one knows the symmetry line, since after a suitable axes rotation the symmetry conditions reduce to a parity verification on the components of the vector field. On the other hand, if such a line is unknown the question is obviously more complex, depending on a free parameter [4,8]. In [7], assuming the system to be analytic and the critical point O to be non-degenerate, it was proved that if a symmetry line exists, then it is contained in the zero-divergence set. This allows for a fast analysis of the possible mirror symmetries, since they are determined by the lowest order non-linear terms of $F(z)$. In [7] this was applied to study quadratic and cubic polynomial systems, also considered in [17].

The existence of a center can also be proved finding a non-linear symmetry. We say that a diffeomorphism $\sigma \in C^1(\Omega, \Omega)$ of Ω onto itself is an *involution* if

$$\sigma^2(z) = \sigma(\sigma(z)) = z \qquad \forall z \in \Omega.$$

An involution σ is said to be a *reversibility* of the system (1) if

$$\phi(t, \sigma(z)) = \sigma(\phi(-t, z)),$$

for all $t \in \mathbb{R}$ and $z \in \Omega$ such that both sides are defined. The non-linear generalization of the symmetry line of a mirror symmetry is the set δ consisting of the σ-fixed points, $\{z \in \Omega : z = \sigma(z)\}$. Poincaré's argument can be easily extended to reversibilities, proving that an orbit intersecting twice δ is a cycle. As a consequence, if δ is a regular curve containing a rotation point of a reversible system, then such a point is a center. Reversibilities can be revealed by examining the symmetry properties of the vector field. In fact, if $\sigma \in C^1(\Omega, \Omega)$, then the σ-reversibility of (1) can be checked by verifying the relationship

$$V(\sigma(z)) = -J_\sigma(z) \cdot V(z), \tag{2}$$

where $J_\sigma(z)$ is the Jacobian matrix of σ at z. Finding a non-linear reversibility is obviously much more difficult than proving mirror symmetry with respect to a line. Moreover the fixed-points set δ of a non-linear reversibility σ is not, in general, a line. The existence of non-linear reversibilities in a neighbourhood of a nilpotent critical point was studied in [16,3,15]. Other results on reversibility, or using reversibility can be found in [1,5,6,10,9,11–13,18].