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Engel theorem through singularities

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ABSTRACT

We prove a singular version of the Engel theorem. We prove a normal form theorem for germs of holomorphic singular Engel systems with good conditions on its singular set. As an application, we prove that there exists an integral analytic curve passing through the singular points of the system. Also, we prove that a globally decomposable Engel system on a four dimensional projective space has singular set with atypical codimension.

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1. Introduction

A germ of *holomorphic Pfaff system* of codimension k on $(\mathbb{C}^n, 0)$ is a subsheaf $\mathcal{I}$ of the cotangent sheaf $\Omega^1_{\mathbb{C}^n}$ of $(\mathbb{C}^n, 0)$ spanned by k germs of holomorphic differential 1-forms $\omega_1, \ldots, \omega_k$ assumed linearly independent at a generic point near 0. We will write $\mathcal{I} = \langle \omega_1, \ldots, \omega_k \rangle$. This system can be represented by the holomorphic k-form $\omega_1 \wedge \ldots \wedge \omega_k$. The singular set of $\mathcal{I}$ is the analytic subset given by

$$\mathrm{Sing}(\mathcal{I}) = \{p \in (\mathbb{C}^n, 0);\ (\omega_1 \wedge \cdots \wedge \omega_k)(p) = 0\}.$$

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Therefore $\mathrm{Sing}(\mathcal{I})$ is defined by $k \times k$ determinants of an $n \times k$ matrix. Therefore, each irreducible component has codimension at most $k+1$. We say that the singular set of $\mathcal{I}$ has *expected codimension* if it is a (startified) submanifold of $\mathbb{C}^n$ of codimension $k+1$.

Let $\mathcal{C} = V(A)$ be a germ of analytic subset in $(\mathbb{C}^n, 0)$ of codimension $\leq k$, with zeros ideal A. If $A = \langle f_1, \dots, f_r \rangle$, then we denote by dA the Pfaff system spanned by $df_1, \dots, df_r$.

By definition, we say that $\mathcal{C} = V(A)$ is an *integral* variety of $\mathcal{I} = \langle \omega_1, \dots, \omega_k \rangle$ if

$$\omega_i \wedge dA \in A \otimes \Omega^{r+1}_{\mathbb{C}^n}, \text{ for each } i = 1, \dots, k.$$

A Pfaff system is called integrable $\mathcal{I}$ if

$$d\mathcal{I} \equiv 0 \mod \mathcal{I}.$$

If $\mathcal{I}$ is integrable, by the classical Frobenius's Theorem, for all points $p \in (\mathbb{C}^n, 0) \backslash \mathrm{Sing}(\mathcal{I})$ there exists an integral complex analytic manifold of codimension k passing through p. In [12] B. Malgrange obtained a Frobenius's Theorem for singular integrable systems with singular set of codimension ≥ 3, showing the existence of integral varieties passing through the singular points of the system.

For a germ of Pfaff system $\mathcal{I}$, we can define its *derived flag* $\mathcal{I}^{(0)} \supset \mathcal{I}^{(1)} \supset \cdots$ by the relations $\mathcal{I}^{(0)} = \mathcal{I}$ and

$$\mathcal{I}^{(i+1)} = \{\alpha \in \mathcal{I}^{(i)} : d\alpha \equiv 0 \mod \mathcal{I}^{(i)}\}.$$

Then, the derived flag of a Pfaff system $\mathcal{I}$ is defined inductively by the exact sequence

$$0 \longrightarrow \mathcal{I}^{(i+1)} \longrightarrow \mathcal{I}^{(i)} \longrightarrow d\mathcal{I}^{(i)} / \left(\mathcal{I}^{(i)} d\mathcal{I}^{(i)}\right) \longrightarrow 0.$$

Since the codimension of each Pfaff system $\mathcal{I}^{(i)}$ is constant on an open and dense set in a neighborhood of 0, then there will be an integer N such that $\mathcal{I}^{(N)} = \mathcal{I}^{(N+1)}$. This integer N is called the *derived length* of $\mathcal{I}$. Note that the Pfaff system $\mathcal{I}^{(N)}$ is always integrable since

$$d\mathcal{I}^{(N)} \equiv 0 \mod \mathcal{I}^{(N)}.$$

If $\mathcal{I}^{(N)} = 0$ we say that the system $\mathcal{I}$ is *completely nonholonomic*. See [3] for more details.

Let Θ_n be the sheaf of germs of holomorphic vector fields on $(\mathbb{C}^n, 0)$. A local (r, n) holomorphic distribution is the germ at $0 \in \mathcal{C}^n$ of a rank r subbundle of the tangent bundle Θ_n

A Pfaff system $\mathcal{I} = \langle \omega_1, \dots, \omega_k \rangle$ of codimension k induces a singular $(n-k, n)$ distribution defined by

$$\mathcal{D}^0 := \mathrm{Ker}(\mathcal{I}) = \{v \in \Theta_n;\ \omega_i(v) \equiv 0,\ \forall\ i\}.$$