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# Serre duality and Hörmander's solution of the $\bar{\partial}$-equation

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ABSTRACT

We use duality in the manner of Serre to generalize a theorem of Hedenmalm on solution of the $\bar{\partial}$ equation with inverse of the weight in Hörmander $L^2$ estimates.



## 1. Introduction

Let $\varphi$ be a $\mathcal{C}^2$ strictly sub harmonic function in the complex plane $\mathbb{C}$, i.e. with $\Delta$ the Laplacian, $\Delta\varphi > 0$ in $\mathbb{C}$. Let $A^2(\mathbb{C}, e^{-2\varphi})$ be the space of all holomorphic functions $g$ in $\mathbb{C}$ such that

$$\|g\|_{L^2(\mathbb{C},e^{-2\varphi})} := \int_{\mathbb{C}} |g|^2 e^{-2\varphi} dA < \infty,$$

with $dA$ the Lebesgue measure in $\mathbb{C}$. Suppose that $f \in L^2(\mathbb{C}, e^{2\varphi})$ verifies

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$$\forall g \in A^2(\mathbb{C}, e^{-2\varphi}), \ \int_{\mathbb{C}} fg dA = 0 \tag{1.1}$$

then in a recent paper H. Hedenmalm [2] proved

**Theorem 1.1.** *Suppose that $f \in L^2(\mathbb{C}, e^{2\varphi})$ verifies condition (1.1) then there exists a solution to the $\bar\partial$-equation $\bar\partial u = f$ with*

$$\int_{\mathbb{C}} |u|^2 e^{2\varphi} \Delta\varphi dA \le \frac{1}{2} \int_{\mathbb{C}} |f|^2 e^{2\varphi} dA.$$

He adds in remark 1.3. that this theorem should generalize to the setting of several complex variables. The aim of this note is to show that he was right.

Let $\varphi$ be a strictly plurisubharmonic function of class $\mathcal{C}^2$ in the Stein manifold $\Omega$. Let $c_\varphi(z)$ be the smallest eigenvalue of $\partial\bar\partial\varphi(z)$, then $\forall z \in \Omega,\ c_\varphi(z) > 0$.

We denote by $L^{2,c}_{p,q}(\Omega, e^\varphi)$ the currents in $L^2_{p,q}(\Omega, e^\varphi)$ with *compact support* in $\Omega$ and $\mathcal{H}_p(\Omega, e^{-\varphi})$ the space of all $(p,\ 0)$, $\bar\partial$ closed forms in $L^2(\Omega,\ e^{-\varphi})$.

If $p = 0$, $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$ is the space of holomorphic functions in $\Omega$. In particular for $\omega \in L^2_{p,n}(\Omega, e^\varphi)$ we set: $\omega \perp \mathcal{H}_{n-p}(\Omega, e^{-\varphi})$ provided that $\forall h \in \mathcal{H}_{n-p}(\Omega, e^{-\varphi})$, $\langle \omega, h\rangle = 0$. This will be more precisely defined in the next section for Stein manifolds.

We shall prove

**Theorem 1.2.** *Let $\Omega$ be a pseudo convex domain in $\mathbb{C}^n$; if $\omega \in L^{2,c}_{p,q}(\Omega,\ e^\varphi)$ with $\bar\partial\omega = 0$ if $q < n$ and $\omega \in L^2_{p,q}(\Omega, e^\varphi)$ with $\omega \perp \mathcal{H}_{n-p}(\Omega,\ e^{-\varphi})$ if $q = n$, then there is $u \in L^2_{p,q-1}(\Omega, c_\varphi e^\varphi)$ such that $\bar\partial u = \omega$, and*

$$\|u\|_{L^2(\Omega, c_\varphi e^\varphi)} \le C\|\omega\|_{L^2(\Omega, e^\varphi)}.$$

Clearly Theorem 1.2 generalizes Hedenmalm's theorem, because in one variable, we have $q = n = 1$ and no compactness assumption on the support of $\omega$ is required.

**Remark 1.3.** A way to see the difference between the cases $q = n$ and $q < n$ is the following.

For $q = n$, $\omega$ is automatically $\bar\partial$-closed, but not automatically the $\bar\partial$ of a rapidly decaying form, which is a reason the orthogonality condition appears. On the other hand, for $q < n$ if $\omega$ is a smooth $\bar\partial$-closed form with compact support, by Andreotti Grauert it is the $\bar\partial$ of another compactly supported form.

In fact, in the case where $\omega \in L^2_{p,q}(\Omega, e^\varphi)$, $\bar\partial\omega = 0$ and $q < n$, I don't know if there is a solution $u \in L^2_{p,q-1}(\Omega, c_\varphi e^\varphi)$ such that $\bar\partial u = \omega$ when $\omega$ is not compactly supported. Never the less I have the feeling that, in the case of $\Omega = \mathbb{C}^n$, this is true.

In the case $\Omega$ is a Stein manifold, the result is more restrictive: we need to take a p.s.h. *exhaustion* function $\varphi$ in $\mathcal{C}^\infty(\Omega)$. These $\varphi$ always exist in a Stein manifold by theorem 5.2.10 in [3].