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Abstract dyadic cubes, maximal operators and Hausdorff content

Hiroki Saito[a,*], Hitoshi Tanaka[b], Toshikazu Watanabe[c]

[a] *Academic support center, Kogakuin University, 2665-1, Nakanomachi, Hachioji-shi, Tokyo, 192-0015, Japan*
[b] *Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo, 153-8914, Japan*
[c] *College of Science and Technology, Nihon University, 1-8-14 Kanda-Surugadai, Chiyoda-ku, Tokyo, 101-8308, Japan*

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ABSTRACT

Let μ be a locally finite Borel measure and $\mathcal{D}$ a family of measurable sets equipped with a certain dyadic structure. For $E \subset \mathbb{R}^n$ and $0 < \alpha \le n$, by α-dimensional Hausdorff content we mean

$$H_\mu^\alpha(E) = \inf \sum_j \mu(Q_j)^{\alpha/n},$$

where the infimum is taken over all coverings of E by countable families of the abstract dyadic cubes $\{Q_j\} \subset \mathcal{D}$. In this paper we study the boundedness of the Hardy–Littlewood maximal operator $M_{\mathcal{D}}^\mu$ adapted to $\mathcal{D}$ and μ, that is, we prove the strong type (p,p) inequality

$$\int (M_{\mathcal{D}}^\mu f)^p \, dH_\mu^\alpha \le \frac{2^{2p+2}}{\min(1,p) - (\alpha/n)} \int |f|^p \, dH_\mu^\alpha$$

for $\alpha/n < p < \infty$, and the weak type $(\alpha/n, \alpha/n)$ inequality

$$H_\mu^\alpha(\{x \in \mathbb{R}^n : M_{\mathcal{D}}^\mu f(x) > t\})$$

* Corresponding author.
E-mail addresses: j1107703@gmail.com (H. Saito), htanaka@ms.u-tokyo.ac.jp (H. Tanaka), twatana@edu.tuis.ac.jp (T. Watanabe).

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$$\leq 4(n/\alpha)^{\alpha/n} t^{-\alpha/n} \int |f|^{\alpha/n}\, dH_\mu^\alpha, \quad t > 0,$$

where the integrals are taken in the Choquet sense.

1. Introduction

It is well known that the Hardy–Littlewood maximal operator and the Hausdorff content are fundamental tools to study Harmonic analysis, potential theory and the theory of partial differential equations; see, for example, the survey [2] and book [3].

For $E \subset \mathbb{R}^n$ and $0 < \alpha \leq n$, let H^α be an α-dimensional Hausdorff content defined by

$$H^\alpha(E) = \inf \sum_j l(Q_j)^\alpha,$$

where the infimum is taken over all coverings of E by countable families of the cubes with sides parallel to the coordinate axes, and $l(Q)$ denotes the side length of the cube Q. In [6], for the Hardy–Littlewood maximal operator M Orobitg and Verdera proved the strong type (p, p) inequality

$$\int (Mf)^p\, dH^\alpha \leq C \int |f|^p\, dH^\alpha \tag{1.1}$$

for $\alpha/n < p < \infty$, and the weak type inequality

$$H^\alpha(\{x : Mf(x) > t\}) \leq C t^{-\alpha/n} \int |f|^{\alpha/n}\, dH^\alpha, \quad t > 0. \tag{1.2}$$

Here, the integrals are taken in the Choquet sense. Previously, Adams proved (1.1) for $p = 1$ and $0 < \alpha < n$ in [1] by using duality of BMO and the Hardy space H^1 among other things. Orobitg and Verdera proved (1.1) and (1.2) directly without duality argument. Recently, Tang [5] proved the boundedness of maximal operators on the weighted Choquet space and the Choquet–Morrey space.

In this paper we investigate dyadic maximal operator and Hausdorff content adapted to an abstract dyadic cubes, and prove these strong and weak type inequalities. To state our results precisely, we will describe some notation and definitions.

Definition 1.1. Let $\mathcal{D}$ be a countable collection of Borel measurable subsets of $\mathbb{R}^n$ with the following properties: