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The weighted composition operators as intertwining operators for holomorphic Lie group representations

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Let D_1 and D_2 be domains in $\mathbb{C}^n$ and let ζ, η be holomorphic functions on D_1 such that $\eta(D_1) \subset D_2$ and $\zeta : D_1 \to \mathbb{C}$. In this paper, we determine necessary and sufficient conditions on ζ, η in order that the weighted composition operator $W_{\zeta,\eta}$ induced by ζ and η be an intertwining operator of holomorphic Lie group representations having the form $(T_g^{(j)} F)(z) = h_g^{(j)}(z) F(k_g^{(j)}(z))$, $j = 1, 2$, where $h_g^{(j)} : D_j \to \mathbb{C}$ and $k_g^{(j)} : D_j \to D_j$ are holomorphic on D_j and g is an element of the Lie group G. Furthermore, we examine conditions on ζ, η to ensure that $W_{\zeta,\eta}$ is also an intertwining operator for the infinitesimal representation of T_g given by

$$\big(\rho^{(j)}(v)F\big)(z) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} T_{g_\epsilon}^{(j)} F(z),$$

where $(g_\epsilon)_{\epsilon\in\mathbb{R}}$ is a smooth one-parameter subgroup of G such that $v = \frac{d}{d\epsilon}|_{\epsilon=0} g_\epsilon$ belongs to the Lie algebra of G.

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0. Introduction

Our project deals with the study of intertwining operators for the square integrable holomorphic Lie group representations as well as for their infinitesimal representations established in [2,7]. Given a Lie group G and a positive real measure μ on a domain D in the Euclidean space $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we denote by $\mathcal{H}(D)$ the vector space of holomorphic functions $F : D \to \mathbb{C}$. If $|F|^2$ is μ-integrable, we denote $F \in \mathcal{H}(D) \cap L^2(\mu)$. The space $\mathcal{H}(D) \cap L^2(\mu)$ with the hermitian form

$$(F_1, F_2)_\mu = \int_D F_1(z)\overline{F_2(z)}\,\mathrm{d}\mu(z) \quad \text{for } F_1, F_2 \in \mathcal{H}(D) \cap L^2(\mu), \tag{0.1}$$

is a Hilbert space. Consider operators $T_g : \mathcal{H}(D) \to \mathcal{H}(D)$ where $g \in G$ and such that

1. $$(T_g F)(z) = h_g(z) F\big(k_g(z)\big), \quad \forall F \in \mathcal{H}(D) \tag{0.2}$$

   where $h_g : D \to \mathbb{C}$ and $k_g : D \to D$ are holomorphic and k_g is an automorphism of D.
2. For $g_1, g_2 \in G$, we have $T_{g_1}(T_{g_2}F) = T_{g_1 g_2}F$.
3. T_g is unitary, if $F \in \mathcal{H}(D) \cap L^2(\mu)$

$$\int_D \big|(T_g F)(z)\big|^2\,\mathrm{d}\mu(z) = \int_D \big|F(z)\big|^2\,\mathrm{d}\mu(z), \quad \forall g \in G. \tag{0.3}$$

Let e denote the neutral element of G then $(T_e F)(z) = F(z)$. Therefore, T_g is invertible with $(T_g)^{-1} = T_{g^{-1}}$. The conditions (1) and (2) together imply

$$h_{g_1}(z) h_{g_2}\big(k_{g_1}(z)\big) = h_{g_1 g_2}(z) \quad \text{and} \quad k_{g_2}\big(k_{g_1}(z)\big) = k_{g_1 g_2}(z). \tag{0.4}$$

When (1), (2), (3) are satisfied, we say that μ is *unitarising* for the representation T_g, and we denote the unitary representation of G by (T_g, μ) (see [2,3,7]). This is the case of (0.5) below. Let $k_g(z) = g^{-1}.z$, where $g.z$ is a left action (respectively $k_g(z) = z.g$ a right action) of G on the domain D and let $k'_g(z)$ be the complex Jacobian of $k_g(z)$. For $\varphi, F \in \mathcal{H}(D)$ and $\alpha \in \mathbb{R}$, we put

$$\big(T_g^{\alpha,\varphi} F\big)(z) = \big(\det\big(k'_g(z)\big)\big)^\alpha e^{\varphi(k_g(z)) - \varphi(z)} F\big(k_g(z)\big). \tag{0.5}$$

We have shown in [2] that if there exists a holomorphic map $\Lambda : D \times D \to \mathbb{C}$ such that $z \to \Lambda(z, \bar{z})$ on D satisfies

$$\big|\det\big(k'_g(z)\big)\big|^{2\alpha-2} \big|\Lambda(z, \bar{z})\big| = \big|\Lambda\big(k_g(z), \overline{k_g(z)}\big)\big|, \quad \forall g \in G,\ \forall z \in D, \tag{0.6}$$