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Maximal multiplier operators in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces ☆

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In this paper we study some estimates of norms in variable exponent Lebesgue spaces for maximal multiplier operators. We will consider the case when multiplier is the Fourier transform of a compactly supported Borel measure.

1. Introduction

Let $f^\wedge(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx$ be a Fourier transform of f. Given a multiplier $m \in L^\infty(\mathbb{R}^n)$, we define the operators M_t, $t > 0$, by $(M_t f)^\wedge(\xi) = \widehat{f}(\xi)m(t\xi)$ and the maximal multiplier operator

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$$\mathcal{M}_m f(x) := \sup_{t>0} |(M_t f)(x)|$$

which is well defined a priori for a Schwartz functions $S(\mathbb{R}^n)$.

It is well known, if multiplier m satisfies the well known Mikhlin–Hörmander condition

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^\alpha$$

for all (or sufficiently large) multiindices α, if $\mathcal{F}^{-1}f(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(x)dx$ is a reverse Fourier transform, then the multiplier operator $f \mapsto \mathcal{F}^{-1}[m\widehat{f}]$ is bounded in $L^p(\mathbb{R}^n)$ when $1 < p < \infty$ (see [12,15,7,11]). Note that maximal operator $\mathcal{M}_m$ formed by multiplier m with the Mikhlin–Hörmander condition in general not bounded on $L^p(\mathbb{R}^n)$. The corresponding example can be found in [4].

We will consider the case when multiplier m is the Fourier transform of a compactly supported Borel measure. In this case the operator M_t, $t > 0$, we can represent as a convolution operator

$$M_t f(x) = \int_S f(x - ty) d\sigma(y),$$

where σ is a compactly supported Borel measure on the set $S \subset \mathbb{R}^n$ and $\widehat{\sigma}(\xi) = m(\xi)$. Obviously we have

$$\mathcal{M}_m f(x) \equiv \mathcal{M}_S f(x) := \sup_{t>0} \left| \int_S f(x - ty) d\sigma(y) \right|.$$

We say that σ is locally uniformly β-dimensional ($\beta > 0$) if $\sigma(\mathrm{B}(x, R)) \le C_\beta R^\beta$, where $\mathrm{B}(x, R)$ is a ball of radius $R \le 1$ centered at x. It is easy to see that a locally uniformly β-dimensional measure must be absolutely continuous with respect to β-dimensional Hausdorff measure μ_β, but such a measure need not exhibit any actual "fractal" behavior. Thus, for example, Lebesgue measure is locally uniformly β-dimensional for any $\beta < n$. We can allow $\beta = 0$ in these definitions, in which case a measure σ is uniformly 0-dimensional if and only if it is finite, and locally uniformly bounded, i.e. $\sigma(B(x, 1))$ is uniformly bounded in x.

Rubio de Francia [16] proved the following

Theorem 1.1. *If $m(\xi)$ is the Fourier transform of a compactly supported Borel measure and satisfies $|m(\xi)| \le (1 + |\xi|)^{-a}$ for some $a > 1/2$ and all $\xi \in \mathbb{R}^n$, then the maximal operator $\mathcal{M}_m$ maps $L^p(\mathbb{R}^n)$ to itself when $p > \frac{2a+1}{2a}$.*

The case when σ is normalized surface measure on the $(n-1)$-dimensional unit sphere was investigated by Stein [17]. According to Stein's theorem for corresponding maximal operator (spherical maximal operator)