Bull. Sci. math. 140 (2016) 112–131

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

Generalised friezes and a modified Caldero–Chapoton map depending on a rigid object, II

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A R T I C L E I N F O

Article history:
Received 27 March 2014
Available online 8 May 2015

MSC:
05E10
13F60
16G70
18E30

Keywords:
Auslander–Reiten triangle
Categorification
Cluster algebra
Cluster category
Cluster tilting object
Rigid object

A B S T R A C T

It is an important aspect of cluster theory that cluster categories are “categorifications” of cluster algebras. This is expressed formally by the (original) Caldero–Chapoton map X which sends certain objects of cluster categories to elements of cluster algebras.

Let $\tau c \to b \to c$ be an Auslander–Reiten triangle. The map X has the salient property that $X(\tau c)X(c) - X(b) = 1$. This is part of the definition of a so-called frieze, see [1].

The construction of X depends on a cluster tilting object. In a previous paper [14], we introduced a modified Caldero–Chapoton map ρ depending on a rigid object; these are more general than cluster tilting objects. The map ρ sends objects of sufficiently nice triangulated categories to integers and has the key property that $\rho(\tau c)\rho(c) - \rho(b)$ is 0 or 1. This is part of the definition of what we call a generalised frieze.

Here we develop the theory further by constructing a modified Caldero–Chapoton map, still depending on a rigid object, which sends objects of sufficiently nice triangulated categories to elements of a commutative ring A. We derive conditions under which the map is a generalised frieze, and show how

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http://dx.doi.org/10.1016/j.bulsci.2015.05.001

the conditions can be satisfied if A is a Laurent polynomial ring over the integers.
The new map is a proper generalisation of the maps X and ρ.

0. Introduction

The (original) Caldero–Chapoton map X is an important object in cluster theory. The arguments of X are certain objects of a cluster category, and the values are the corresponding elements of a cluster algebra. The map X expresses that the cluster category is a categorification of the cluster algebra, see [7,9,10,13,17]. For example, Fig. 1 shows the Auslander–Reiten (AR) quiver of $\mathsf{C}(A_5)$, the cluster category of Dynkin type A_5, with a useful "coordinate system". Fig. 2 shows the AR quiver again, with the values of X on the indecomposable objects of $\mathsf{C}(A_5)$. The values are Laurent polynomials over $\mathbb{Z}$; indeed, the cluster algebra consists of such Laurent polynomials.

It is a salient property of X that it is a *frieze* in the sense of [1], that is, if $\tau c \to b \to c$ is an AR triangle then

$$X(\tau c)X(c) - X(b) = 1,$$

see [12, Theorem] and [7, Prop. 3.10]. In the case of $\mathsf{C}(A_5)$, this means that for each "diamond" in the AR quiver, of the form

$$\begin{tikzcd}[row sep=small, column sep=small] & b_1 \arrow[dr] & \\ \tau c \arrow[ur] \arrow[dr] & & c, \\ & b_2 \arrow[ur] & \end{tikzcd} \tag{0.1}$$

we have

$$X(\tau c)X(c) - X(b_1)X(b_2) = 1.$$

The definition of X depends on a cluster tilting object T. For instance, the X shown in Fig. 2 depends on the T which has the indecomposable summands shown by red and blue vertices in Fig. 1.

This paper is about a modified Caldero–Chapoton map ρ which is more general than X in two respects: it depends on a rigid object R and has values in a general commutative ring A. An object R is rigid if $\mathrm{Hom}(R, \Sigma R) = 0$. This is much weaker than being cluster tilting: recall that T is cluster tilting if $\mathrm{Hom}(T, \Sigma t) = 0 \Leftrightarrow t \in \mathrm{add}\, T \Leftrightarrow \mathrm{Hom}(t, \Sigma T) = 0$. Our first main result gives conditions under which ρ is a *generalised frieze*, in the sense that if $\tau c \to b \to c$ is an AR triangle then