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An approach to finite-dimensional real division composition algebras through reflections

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A B S T R A C T

We consider the category of all finite-dimensional real composition algebras which are division algebras. These are precisely the finite-dimensional absolute valued algebras, and exist only in dimension 1, 2, 4 and 8.

We construct three decompositions of this category, each determined by the number of reflections composing left and right multiplication by idempotents. As a consequence, we obtain new full subcategories in dimension 8, in which all morphisms are automorphisms of the octonions. This reduces considerable parts of the still open classification problem in dimension 8 to the normal form problem of an action of the automorphism group of the octonions, which is a compact Lie group of type G_2, on pairs of orthogonal maps. We describe these subcategories further in terms of subgroups of $\mathrm{Aut}(\mathbb{O})$ and their cosets, which we express geometrically.

This extends the study of finite-dimensional real division composition algebras with a one-sided unity.

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1. Introduction

An *algebra* over a field k is a vector space A over k equipped with a k-bilinear multiplication $A \times A \to A, (x, y) \mapsto xy$. In general, we do not assume commutativity, associativity, or the existence of a unity. A *composition algebra* is a non-zero algebra endowed with a non-degenerate multiplicative quadratic form, while a *division algebra* is a non-zero algebra in which left and right multiplication by any non-zero element is bijective. In this article, we will study finite-dimensional real algebras which are both division and composition algebras; these are precisely the finite-dimensional absolute valued algebras. A real algebra is *absolute valued* if it is non-zero and endowed with a multiplicative norm. It is known that the norm $\|\cdot\|$ and the quadratic form q of any finite-dimensional absolute valued algebra A are uniquely determined by the multiplication, and that $q(x) = \|x\|^2$ for any $x \in A$. This allows us to speak non-ambiguously of orthogonality and isometry. If A is moreover unital, it is equipped with the *standard involution* $\kappa = \kappa_A : x \mapsto \bar{x}$, defined by linearly extending $\bar{1} = 1$ and $\bar{x} = -x$ whenever $x \perp 1$.

We write $\mathcal{A}$ for the category of finite-dimensional absolute valued algebras, where the morphisms are the non-zero algebra homomorphisms. It is known that these are isometries (by the aforementioned uniqueness of the norm). Finite-dimensional absolute valued algebras exist only in dimension 1, 2, 4 and 8. This was proven by Albert in [1] from 1947, by expressing the algebras as isotopes of one of the four classical unital division algebras of the real numbers, complex numbers, quaternions and octonions, each endowed with its Euclidean norm. Namely, he proved the following.

Proposition 1.1. *Every finite-dimensional absolute valued algebra is isomorphic to an orthogonal isotope $A = \mathbb{A}_{f,g}$ of a unique $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, i.e. $A = \mathbb{A}$ as a vector space, and the multiplication $\cdot$ in A is given by*

$$x \cdot y = f(x)g(y)$$

for all $x, y \in A$, where f and g are linear orthogonal operators on $\mathbb{A}$, and multiplication in $\mathbb{A}$ is written as juxtaposition. Moreover, the norms of A and $\mathbb{A}$ coincide.

In dimension at most four, a complete classification has been achieved. Conditions for when two eight-dimensional absolute valued algebras are isomorphic were obtained in [4], and this was formalized as a description (in the sense of Dieterich) in [3]. These results are reviewed in Section 2. The conditions use the *Principle of triality* due to Elie Cartan in [5]. The classification problem, however, remains unsolved in dimension eight and has proven hard. Indeed, Albert's result parameterizes eight-dimensional absolute valued algebras using the product of two copies of the orthogonal group on $\mathbb{R}^8$, which, as a topological manifold, has dimension 56. Furthermore, using the Principle of triality explicitly involves technical difficulties.

Orthogonal maps on d-dimensional Euclidean space enjoy the property that they are composed of at most d hyperplane reflections. This is asserted in the well-known