Bull. Sci. math. 139 (2015) 923–936

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

Formulas for phase recovering from phaseless scattering data at fixed frequency

R.G. Novikov [a,b,*]

[a] CNRS (UMR 7641), Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau, France
[b] IEPT RAS, 117997 Moscow, Russia

ARTICLE INFO

Article history:
Received 7 February 2015
Available online 17 April 2015

MSC:
35J10
35P25
35R30
81U40

Keywords:
Schrödinger equation
Monochromatic scattering data
Phase recovering
Phaseless inverse scattering

ABSTRACT

We consider quantum and acoustic wave propagation at fixed frequency for compactly supported scatterers in dimension $d \geq 2$. In these framework we give explicit formulas for phase recovering from appropriate phaseless scattering data. As a corollary, we give global uniqueness results for quantum and acoustic inverse scattering at fixed frequency without phase information.

1. Introduction

We consider the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \tag{1.1}$$

* Correspondence to: CNRS (UMR 7641), Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau, France.

E-mail address: novikov@cmap.polytechnique.fr.

http://dx.doi.org/10.1016/j.bulsci.2015.04.005

where Δ is the Laplacian, v is a scalar potential such that

$$v \in L^\infty(\mathbb{R}^d), \quad supp\, v \subset D,$$
$$D \text{ is an open bounded domain in } \mathbb{R}^d. \tag{1.2}$$

Eq. (1.1) can be considered as the quantum mechanical Schrödinger equation at fixed energy E.

Eq. (1.1) can also be considered as the acoustic equation at fixed frequency ω. In this setting

$$E = \left(\frac{\omega}{c_0}\right)^2, \quad v(x) = (1 - n^2(x))\left(\frac{\omega}{c_0}\right)^2, \tag{1.3}$$

where c_0 is a reference sound speed, $n(x)$ is a scalar index of refraction.

For Eq. (1.1) we consider the classical scattering solutions ψ^+ continuous and bounded on $\mathbb{R}^d$ and specified by the following asymptotics as $|x| \to \infty$:

$$\psi^+(x,k) = e^{ikx} + c(d,|k|)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f\left(k, |k|\frac{x}{|x|}\right) + O\left(\frac{1}{|x|^{(d+1)/2}}\right),$$
$$x \in \mathbb{R}^d,\ k \in \mathbb{R}^d,\ k^2 = E,\ c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}, \tag{1.4}$$

where a priori unknown function $f = f(k,l)$, $k,l \in \mathbb{R}^d$, $k^2 = l^2 = E$, arising in (1.4) is the classical scattering amplitude for (1.1).

In order to find ψ^+ and f from v one can use the following Lippmann–Schwinger integral equation (1.5) and formula (1.7) (see, e.g., [4,10]):

$$\psi^+(x,k) = e^{ikx} + \int_D G^+(x-y,k)v(y)\psi^+(y,k)dy, \tag{1.5}$$

$$G^+(x,k) \stackrel{\text{def}}{=} -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0} = G_0^+(|x|,|k|), \tag{1.6}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E$, and G_0^+ depends also on d;

$$f(k,l) = (2\pi)^{-d} \int_D e^{-ily} v(y)\psi^+(y,k)dy, \tag{1.7}$$

where $k \in \mathbb{R}^d$, $l \in \mathbb{R}^d$, $k^2 = l^2 = E$.

We recall that ψ^+ describes scattering of the incident plane waves e^{ikx} on the potential v. And the second term of the right-hand side of (1.4) describes the scattered spherical waves.

In addition to ψ^+, we consider also the function R^+ describing scattering of spherical waves generated by point sources. The function $R^+ = R^+(x,x',E)$, $x \in \mathbb{R}^d$, $x' \in \mathbb{R}^d$, can be defined as the Schwartz kernel of the standard resolvent $(-\Delta + v - E - i0)^{-1}$. Note