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Bifurcation and multiplicity results for critical nonlocal fractional Laplacian problems ☆

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A B S T R A C T

In this paper we consider the following critical nonlocal problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where $s \in (0,1)$, Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with continuous boundary, λ is a positive real parameter, $2^* := 2n/(n-2s)$ is the fractional critical Sobolev exponent, while $\mathcal{L}_K$ is the nonlocal integrodifferential operator

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)\, dy\,, \quad x \in \mathbb{R}^n\,,$$

whose model is given by the fractional Laplacian $-(-\Delta)^s$.

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Along the paper, we prove a multiplicity and bifurcation result for this problem, using a classical theorem in critical points theory. Precisely, we show that in a suitable left neighborhood of any eigenvalue of $-\mathcal{L}_K$ (with Dirichlet boundary data) the number of nontrivial solutions for the problem under consideration is at least twice the multiplicity of the eigenvalue. Hence, we extend the result got by Cerami, Fortunato and Struwe in [14] for classical elliptic equations, to the case of nonlocal fractional operators.

1. Introduction

In recent years, nonlocal problems and operators have been widely studied in the literature and have attracted the attention of lot of mathematicians coming from different research areas. The interest towards equations involving nonlocal operators has grown more and more, thanks to their intriguing analytical structure and in view of several applications in a wide range of contexts. Indeed, fractional and nonlocal operators appear in concrete applications in many fields such as, among the others, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, thin obstacle problem, optimal transport, image reconstruction, through a new and fascinating scientific approach (see, e.g., the papers [2,8,11,12,18,23,37–39] and the references therein).

After the seminal paper [10] by Brezis and Nirenberg, the critical problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2_*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

has been widely studied in the literature (see, e.g. [13–17,19,20,22,26,35,40] just to name a few), also due to its relevant relations with problems arising in differential geometry and in physics, where a lack of compactness occurs (see, for instance, the famous Yamabe problem). Here Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2$, and $2_* := 2n/(n-2)$ is the critical Sobolev exponent.

The first multiplicity result for problem (1.1) was proved by Cerami, Fortunato and Struwe in [14], where it was shown that in a suitable left neighborhood of any eigenvalue of $-\Delta$ (with Dirichlet boundary data) the number of solutions is at least twice the multiplicity of the eigenvalue. The authors also gave an estimate of the length of this neighborhood, which depends on the best critical Sobolev constant, on the Lebesgue measure of the set where the problem is set and on the space dimension.

Later, in [15] the authors proved that in dimension $n \geqslant 6$ and for $\lambda > 0$ less than the first eigenvalue of $-\Delta$ (with homogeneous Dirichlet boundary conditions), problem (1.1)