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Pseudodifferential extensions and adiabatic deformation of smooth groupoid actions

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The adiabatic groupoid $\mathcal{G}_{ad}$ of a smooth groupoid $\mathcal{G}$ is a deformation relating $\mathcal{G}$ with its algebroid. In a previous work, we constructed a natural action of $\mathbb{R}$ on the C*-algebra of zero order pseudodifferential operators on $\mathcal{G}$ and identified the crossed product with a natural ideal $J(\mathcal{G})$ of $C^*(\mathcal{G}_{ad})$. In the present paper we show that $C^*(\mathcal{G}_{ad})$ itself is a pseudodifferential extension of this crossed product in a sense introduced by Saad Baaj. Let us point out that we prove our results in a slightly more general situation: the smooth groupoid $\mathcal{G}$ is assumed to act on a C*-algebra A. We construct in this generalized setting the extension of order 0 pseudodifferential operators $\Psi(A,\mathcal{G})$ of the associated crossed product $A \rtimes \mathcal{G}$. We show that $\mathbb{R}$ acts naturally on $\Psi(A,\mathcal{G})$ and identify the crossed product of A by the action of the adiabatic groupoid $\mathcal{G}_{ad}$ with an extension of the crossed product $\Psi(A,\mathcal{G}) \rtimes \mathbb{R}$. Note that our construction of $\Psi(A,\mathcal{G})$ unifies the ones of Connes (case $A = \mathbb{C}$) and of Baaj ($\mathcal{G}$ is a Lie group).

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1. Introduction

Alain Connes in [7, Chapter VIII] pointed out that smooth groupoids offer a perfect setting for index theory. Since then, this fact has been explored and exploited by Connes as well as many other authors, in many geometric situations (see [11] for a review).

In [10, Section II.5], A. Connes constructed a beautiful groupoid, which he called the "tangent groupoid", that interpolates between the pair groupoid $M \times M$ of a (smooth, compact) manifold M and the tangent bundle TM of M. He showed that this groupoid describes the analytic index on M in a way not involving (pseudo)differential operators at all, and gave a proof of the Atiyah–Singer Index Theorem based on this groupoid.

This idea of a deformation groupoid was then used in [15, Section III], and extended in [22,23] to the general case of a smooth groupoid, where the authors associated to every smooth groupoid $\mathcal{G}$ an *adiabatic groupoid* $\mathcal{G}_{ad}$, which is obtained by applying the "deformation to the normal cone" construction to the inclusion $\mathcal{G}^{(0)} \to \mathcal{G}$ of the unit space of $\mathcal{G}$ into $\mathcal{G}$. Moreover, it was shown in [22, Théorème 2.1] that this adiabatic groupoid still describes the analytic index of the groupoid $\mathcal{G}$ in this generalized situation.

In [12], we further explored the relationship between pseudodifferential calculus on $\mathcal{G}$ and its adiabatic deformation $\mathcal{G}_{ad}$. An ideal $J(\mathcal{G}) \subset C^*(\mathcal{G}_{ad})$ which sits in an exact sequence $0 \to J(\mathcal{G}) \to C^*(\mathcal{G}_{ad}) \to C(\mathcal{G}^{(0)}) \to 0$ plays a crucial role in our constructions. We construct a canonical Morita equivalence between the algebra $\Psi^*(\mathcal{G})$ of order 0 pseudodifferential operators on $\mathcal{G}$ and the crossed product $J(\mathcal{G}) \rtimes \mathbb{R}_+^*$ of $J(\mathcal{G})$ by the natural action of $\mathbb{R}_+^*$.

It appeared that $J(\mathcal{G})$ is canonically isomorphic to the crossed product $\Psi^*(\mathcal{G}) \rtimes \mathbb{R}$ associated with a natural action of $\mathbb{R}$ on the algebra $\Psi^*(\mathcal{G})$. A natural question is then: can one recognize the C^*-algebra $C^*(\mathcal{G}_{ad})$ in these terms?

In the present paper, we answer this question, thanks to [3,4], where Baaj constructed an extension of pseudodifferential operators of order 0 of the crossed product of a C^*-algebra A by the action of a Lie group H – with Lie algebra $\mathfrak{H}$. Denote by $S^*\mathfrak{H}$ the sphere in $\mathfrak{H}^*$. Baaj's exact sequence reads

$$0 \to A \rtimes H \to \Psi_0^*(A, H) \xrightarrow{\sigma} C\big(S^*\mathfrak{H}\big) \otimes A \to 0.$$

Let $\mu : C(\mathcal{G}^{(0)}) \to \Psi^*(\mathcal{G})$ be the inclusion by multiplication operators. In the present paper, we construct a commutative diagram, whose first line is Baaj's exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Psi^*(\mathcal{G}) \rtimes \mathbb{R} & \longrightarrow & \Psi_0^*(\Psi^*(\mathcal{G}), \mathbb{R}) & \xrightarrow{\sigma} & \Psi^*(\mathcal{G}) \oplus \Psi^*(\mathcal{G}) & \longrightarrow & 0 \\ & & \uparrow \simeq & & \uparrow & & \uparrow \mu_0 & & \\ 0 & \longrightarrow & J(\mathcal{G}) & \longrightarrow & C^*(\mathcal{G}_{ad}) & \longrightarrow & C(\mathcal{G}^{(0)}) & \longrightarrow & 0 \end{array} \tag{1}$$

where $\mu_0(f) = (\mu(f), 0)$.