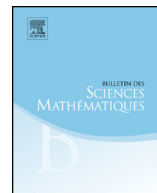




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# Pseudodifferential extensions and adiabatic deformation of smooth groupoid actions



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## ABSTRACT

The adiabatic groupoid  $\mathcal{G}_{ad}$  of a smooth groupoid  $\mathcal{G}$  is a deformation relating  $\mathcal{G}$  with its algebroid. In a previous work, we constructed a natural action of  $\mathbb{R}$  on the  $C^*$ -algebra of zero order pseudodifferential operators on  $\mathcal{G}$  and identified the crossed product with a natural ideal  $J(\mathcal{G})$  of  $C^*(\mathcal{G}_{ad})$ . In the present paper we show that  $C^*(\mathcal{G}_{ad})$  itself is a pseudodifferential extension of this crossed product in a sense introduced by Saad Baaj. Let us point out that we prove our results in a slightly more general situation: the smooth groupoid  $\mathcal{G}$  is assumed to act on a  $C^*$ -algebra  $A$ . We construct in this generalized setting the extension of order 0 pseudodifferential operators  $\Psi(A, \mathcal{G})$  of the associated crossed product  $A \rtimes \mathcal{G}$ . We show that  $\mathbb{R}$  acts naturally on  $\Psi(A, \mathcal{G})$  and identify the crossed product of  $A$  by the action of the adiabatic groupoid  $\mathcal{G}_{ad}$  with an extension of the crossed product  $\Psi(A, \mathcal{G}) \rtimes \mathbb{R}$ . Note that our construction of  $\Psi(A, \mathcal{G})$  unifies the ones of Connes (case  $A = \mathbb{C}$ ) and of Baaj ( $\mathcal{G}$  is a Lie group).

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## 1. Introduction

Alain Connes in [7, Chapter VIII] pointed out that smooth groupoids offer a perfect setting for index theory. Since then, this fact has been explored and exploited by Connes as well as many other authors, in many geometric situations (see [11] for a review).

In [10, Section II.5], A. Connes constructed a beautiful groupoid, which he called the “tangent groupoid”, that interpolates between the pair groupoid  $M \times M$  of a (smooth, compact) manifold  $M$  and the tangent bundle  $TM$  of  $M$ . He showed that this groupoid describes the analytic index on  $M$  in a way not involving (pseudo)differential operators at all, and gave a proof of the Atiyah–Singer Index Theorem based on this groupoid.

This idea of a deformation groupoid was then used in [15, Section III], and extended in [22,23] to the general case of a smooth groupoid, where the authors associated to every smooth groupoid  $\mathcal{G}$  an *adiabatic groupoid*  $\mathcal{G}_{ad}$ , which is obtained by applying the “deformation to the normal cone” construction to the inclusion  $\mathcal{G}^{(0)} \rightarrow \mathcal{G}$  of the unit space of  $\mathcal{G}$  into  $\mathcal{G}$ . Moreover, it was shown in [22, Théorème 2.1] that this adiabatic groupoid still describes the analytic index of the groupoid  $\mathcal{G}$  in this generalized situation.

In [12], we further explored the relationship between pseudodifferential calculus on  $\mathcal{G}$  and its adiabatic deformation  $\mathcal{G}_{ad}$ . An ideal  $J(\mathcal{G}) \subset C^*(\mathcal{G}_{ad})$  which sits in an exact sequence  $0 \rightarrow J(\mathcal{G}) \rightarrow C^*(\mathcal{G}_{ad}) \rightarrow C(\mathcal{G}^{(0)}) \rightarrow 0$  plays a crucial role in our constructions. We construct a canonical Morita equivalence between the algebra  $\Psi^*(\mathcal{G})$  of order 0 pseudodifferential operators on  $\mathcal{G}$  and the crossed product  $J(\mathcal{G}) \rtimes \mathbb{R}_+^*$  of  $J(\mathcal{G})$  by the natural action of  $\mathbb{R}_+^*$ .

It appeared that  $J(\mathcal{G})$  is canonically isomorphic to the crossed product  $\Psi^*(\mathcal{G}) \rtimes \mathbb{R}$  associated with a natural action of  $\mathbb{R}$  on the algebra  $\Psi^*(\mathcal{G})$ . A natural question is then: can one recognize the  $C^*$ -algebra  $C^*(\mathcal{G}_{ad})$  in these terms?

In the present paper, we answer this question, thanks to [3,4], where Baaj constructed an extension of pseudodifferential operators of order 0 of the crossed product of a  $C^*$ -algebra  $A$  by the action of a Lie group  $H$  – with Lie algebra  $\mathfrak{h}$ . Denote by  $S^*\mathfrak{h}$  the sphere in  $\mathfrak{h}^*$ . Baaj’s exact sequence reads

$$0 \rightarrow A \rtimes H \rightarrow \Psi_0^*(A, H) \xrightarrow{\sigma} C(S^*\mathfrak{h}) \otimes A \rightarrow 0.$$

Let  $\mu: C(\mathcal{G}^{(0)}) \rightarrow \Psi^*(\mathcal{G})$  be the inclusion by multiplication operators. In the present paper, we construct a commutative diagram, whose first line is Baaj’s exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Psi^*(\mathcal{G}) \rtimes \mathbb{R} & \longrightarrow & \Psi_0^*(\Psi^*(\mathcal{G}), \mathbb{R}) & \xrightarrow{\sigma} & \Psi^*(\mathcal{G}) \oplus \Psi^*(\mathcal{G}) \longrightarrow 0 \\ & & \uparrow \simeq & & \uparrow & & \uparrow \mu_0 \\ 0 & \longrightarrow & J(\mathcal{G}) & \longrightarrow & C^*(\mathcal{G}_{ad}) & \longrightarrow & C(\mathcal{G}^{(0)}) \longrightarrow 0 \end{array} \quad (1)$$

where  $\mu_0(f) = (\mu(f), 0)$ .

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