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Exponential convergence to equilibrium for the homogeneous Landau equation with hard potentials

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A B S T R A C T

This paper deals with the long time behaviour of solutions to the spatially homogeneous Landau equation with hard potentials. We prove an exponential in time convergence towards the equilibrium with the optimal rate given by the spectral gap of the associated linearised operator. This result improves the polynomial in time convergence obtained by Desvillettes and Villani [5]. Our approach is based on new decay estimates for the semigroup generated by the linearised Landau operator in weighted (polynomial or stretched exponential) L^p-spaces, using a method developed by Gualdani, Mischler and Mouhot [7].

1. Introduction and main results

This work deals with the asymptotic behaviour of solutions to the spatially homogeneous Landau equation for hard potentials. It is well known that these solutions converge towards the Maxwellian equilibrium when time goes to infinity and we are interested in quantitative rates of convergence.

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On the one hand, in the case of Maxwellian molecules, Villani [15] and Desvillettes and Villani [5] have proved a linear functional inequality between the entropy and entropy dissipation by constructive methods, from which one deduces an exponential convergence (with quantitative rate) of the solution to the Landau equation towards the Maxwellian equilibrium in relative entropy, which in turn implies an exponential convergence in L^1-distance (thanks to the Csiszár–Kullback–Pinsker inequality). This kind of linear functional inequality relating entropy and entropy dissipation is known as Cercignani's Conjecture in Boltzmann and Landau theory, for more details and a review of results we refer to [3].

On the other hand, in the case of hard potentials, Desvillettes and Villani [5] have proved a functional inequality for entropy–entropy dissipation that is not linear, from which one obtains a polynomial in time convergence of solutions towards the equilibrium, again in relative entropy, which implies the same type of convergence in L^1-distance.

Before going further on details of existing results and on the contributions of the present work, we shall introduce in a precise manner the problem addressed here. In kinetic theory, the Landau equation is a model in plasma physics that describes the evolution of the density in the phase space of all positions and velocities of particles. Assuming that the density function does not depend on the position, we obtain the *spatially homogeneous Landau equation* in the form

$$\begin{cases} \partial_t f = Q(f,f) \\ f_{|t=0} = f_0, \end{cases} \tag{1.1}$$

where $f = f(t,v) \geq 0$ is the density of particles with velocity v at time t, $v \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$. The Landau operator Q is a bilinear operator given by

$$Q(g,f) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*)[g_* \partial_j f - f \partial_j g_*] \, dv_*, \tag{1.2}$$

where here and below we shall use the convention of implicit summation over repeated indices and we use the shorthand $g_* = g(v_*)$, $\partial_j g_* = \partial_{v_{*j}} g(v_*)$, $f = f(v)$ and $\partial_j f = \partial_{v_j} f(v)$.

The matrix a is nonnegative, symmetric and depends on the interaction between particles. If two particles interact with a potential proportional to $1/r^s$, where r denotes their distance, a is given by (see for instance [16])

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \tag{1.3}$$

with $\gamma = (s-4)/s$. We usually call hard potentials if $\gamma \in (0,1]$, Maxwellian molecules if $\gamma = 0$, soft potentials if $\gamma \in (-3,0)$ and Coulombian potential if $\gamma = -3$. Through this paper we shall consider the case of hard potentials $\gamma \in (0,1]$.